AST 376 Cosmology — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith
(Dated: February 4, 2014)

GENERAL RELATIVITY (GR) – A VERY BRIEF INTRODUCTION

So far we have pushed Newtonian cosmology to the limit; we have gone as far as we can without appealing to general relativity (GR). Some of the important topics we have discussed are the cosmological principle and the dichotomy of an inwardly-directed gravitational force competing against the impetus of Hubble flow. However, Einstein-de Sitter is as far as we can go with Newtonian cosmology. Recall the critical density is $\rho_{\rm crit,0} \equiv 3H_0^2/8\pi G$ so if $\rho > \rho_{\rm crit,0}$ we have a Big Crunch and if $\rho \leq \rho_{\rm crit,0}$ the universe expands forever. An Einstein-de Sitter universe has $\rho = \rho_{\rm crit,0}$, is flat (i.e. $\Omega = 1$), and is matter-dominated so that

$$\boxed{H = H_0 a^{-3/2} \qquad \text{yielding the solution} \qquad a(t) \propto t^{2/3} \qquad (\text{“Einstein-de Sitter universe”})\,.} \tag{1}$$

Special Relativity (SR)

We first cover a few essential concepts and formalisms that are most easily introduced in special relativity (SR). Events A and B can be described in different (x-directed) inertial reference frames by generalized 4-vectors. Here we use a prime to denote a different coordinate system, however, both are valid descriptions of what is happening in spacetime:

$$A = (t_{\rm A}, x_{\rm A}, y_{\rm A}, z_{\rm A}) = (t'_{\rm A}, x'_{\rm A}, y'_{\rm A}, z'_{\rm A}) = A'$$
and
$$B = (t_{\rm B}, x_{\rm B}, y_{\rm B}, z_{\rm B}) = (t'_{\rm B}, x'_{\rm B}, y'_{\rm B}, z'_{\rm B}) = B'\,.$$

We may compare two events by their differences, which signify the physical length or elapsed time:

$$\Delta t = t_{\rm B} - t_{\rm A}\,, \qquad \Delta x = x_{\rm B} - x_{\rm A}\,, \qquad \Delta t' = t'_{\rm B} - t'_{\rm A}\,, \qquad \Delta x' = x'_{\rm B} - x'_{\rm A}\,, \qquad \text{etc.}$$

However, because time is fundamentally different than space we can no longer apply a simple Galilean coordinate transformation (e.g. $x' = x - vt$). Instead we use a **Lorentz transformation** which acts as a dictionary to communicate the events from the frames of two observers:

$$\Delta t' = \frac{\Delta t - (v/c^2)\Delta x}{\sqrt{1-(v/c)^2}}\,, \qquad \Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1-(v/c)^2}}\,, \qquad \Delta y' = \Delta y\,, \quad \text{and} \qquad \Delta z' = \Delta z\,. \tag{2}$$

Note: In SR we often use $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$ to simplify our notation.
The differences can be made arbitrarily small so that $\Delta x \to dx$. Thus, the spacetime interval corresponding to this Lorentz transformation is

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \qquad (\text{“invariant spacetime interval”})\,.} \tag{3}$$

Note: We could have chosen to have $d\tilde{s}^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ instead but as we prefer to think of positive cosmological distances we have $(+)$ for spacelike vectors and $(-)$ for timelike vectors. The spacetime interval gives us access to what is “real” in the universe.

Now the **proper time** τ is always measured in the rest-frame where the spatial coordinates do not change (i.e. $d\tau = dt'$ and $dx' = dy' = dz' = 0$). In this frame the spacetime interval is simplified:

$$\boxed{ds^2 = -c^2 d\tau^2\,.} \tag{4}$$

Likewise, the **proper distance** ℓ is the at rest distance where the time is fixed (i.e. $dt = 0$), or

$$\boxed{ds^2 = d\ell^2 = dx^2 + dy^2 + dz^2\,.} \tag{5}$$

We can write the spacetime interval in a neat (compact) way by viewing it as a matrix multiplication. If we represent the **Minkowski metric** $\eta_{\mu\nu}$ by the diagonal matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,$$

and a general differential distance as the vector $dx^\mu = (cdt, dx, dy, dz)$ then the spacetime interval can be written as

$$ds^2 = \sum_\mu \sum_\nu \eta_{\mu\nu} dx^\mu dx^\nu = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} = -c^2 dt^2 + dx^2 + dy^2 + dz^2\,.$$

An even more economical expression uses the "Einstein summation convention" where we sum over repeated indices which will be important in GR so get used to the following(!):

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad \text{(Minkowski Line Element).}} \tag{6}$$

In SR spacetime is said to be "flat" so what does this mean? For our purposes we say the metric can be transformed so that the derivatives of the metric vanish (i.e. $\partial \eta_{\mu\nu}/\partial x^\alpha = 0$). Note however that different coordinate systems could be more complicated. For example, the Minkowski metric in spherical coordinates is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin\theta)$. In GR we divorce the coordinate system from the physics! True physics originates from proper time, not coordinate time.

We conclude the lecture by writing the (SR) laws of motion in a coordinate-invariant way. We have already seen this because the spacetime interval is a Lorentz invariant quantity. There are others! For example, velocity is defined according to the proper time as

$$v^\alpha = \frac{dx^\alpha}{d\tau} = \gamma \frac{dx^\alpha}{dt} = \gamma\,(c, \vec{v})\;, \tag{7}$$

where $\vec{v}$ is the 3-dimensional coordinate velocity we are used to. Likewise the "4-momentum" is

$$p^\alpha = m_0 v^\alpha = \left(\frac{\epsilon}{c}, \vec{p}\right)\;, \tag{8}$$

where m_0 is the rest mass, ϵ is the energy, and $\vec{p}$ is the familiar 3-dimensional momentum. Now we have a method to write the laws of nature in an invariant form:

If Newton says

$$\vec{F} = m_0 \vec{a}\,,$$

then Einstein says

$$f^\alpha = \frac{dp^\alpha}{d\tau}\,.$$

Force-free motion requires $dp^\alpha/d\tau = 0$.

Note: The equivalent of an 'inertial' frame in SR is a 'freely-falling' frame in GR.

Newtonian Gravity

Recall the universal law of gravitation for two particles:

$$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r \quad \text{or} \quad \vec{F} = m\vec{g}\,.$$

However, the mass here is the ‘gravitational mass’ m_g because it measures how strongly the object is coupled to the gravitational field.

We may introduce this in terms of a gravitational potential $\varphi = -GM/r$ for a point source:

$$\boxed{\vec{g} = -\vec{\nabla}\varphi\,.} \tag{9}$$

Or re-phrase it by taking the divergence and identifying the Laplacian operator ($\nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla}$) so

$$\vec{\nabla}\cdot\vec{g} = -\nabla^2\phi\,.$$

If we integrate over a surface specified by $d\vec{A}$ and apply Gauss’s Theorem then

$$-\int \nabla^2\varphi dV = \int \vec{\nabla}\cdot\vec{g}dV = \int \vec{g}\cdot d\vec{A} = -4\pi r^2 g = -4\pi G\int \rho(r)dV$$

and we get Poisson’s equation:

$$\boxed{\nabla^2\varphi = 4\pi G\rho \qquad \text{(Poisson's Equation)}\,.} \tag{10}$$

This is the “field equation for Newtonian gravity” and gives an equation for the field for all $\vec{r}$.

The Equation of motion is given by applying Newton’s Second Law, where the mass for this law is the ‘inertial mass’ m_i,

$$m_i\frac{d^2\vec{r}}{dt^2} = \vec{F} = m_g\vec{g}\,,$$

and it is not at all obvious that $m_i = m_g = m$! We shall take this as an experimental fact as shown by Galileo at the tower in Pisa. (We shall see that Einstein’s theory predicts the equivalence of the different types of masses.) The next step is to cancel the masses to get the equation of motion

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\vec{\nabla}\phi \qquad \text{(Equation of Motion)}\,.} \tag{11}$$