AST 376 Cosmology — Lecture Notes

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DARK ENERGY

We have now come to the most curious constituent of the cosmic energy budget – dark energy.

Basic Properties

The physics of "PdV work" reveals the first mysterious property of dark energy. Consider a moving piston. For small changes in volume the work done by expanding the gas is $\Delta W = -P\Delta V$.

Q: Where does this work come from?
A: It is removed from the internal (or thermal) energy of the gas.

For now assume the vacuum energy density is constant, i.e. the energy only depends on volume:

$$\rho_{\text{vac}} = \text{constant} \qquad \Rightarrow \qquad \Delta E_{\text{vac}} = \rho_{\text{vac}} c^2 \Delta V \,.$$

Conservation of energy requires the energy difference balances the work exerted:

$$\Delta E_{\text{vac}} = \Delta W \qquad \Rightarrow \qquad \rho_{\text{vac}} c^2 \Delta V = -P_{\text{vac}} \Delta V \,.$$

Specifically, this means the pressure is negative, called a *tension* in analogy to a spring,

$$\boxed{P_{\text{vac}} = -\rho_{\text{vac}} c^2 \qquad (\text{“Negative vacuum pressure”})\,.} \tag{1}$$

In general we write the "equation of state" (e.o.s.) of dark energy as

$$\boxed{P_{\text{vac}} = w\rho_{\text{vac}} c^2 \qquad (\text{“Vacuum e.o.s.”})\,,} \tag{2}$$

where w may be time dependent, i.e. $w = w(z)$.
Note: The e.o.s. depends on the material, e.g. $P = nk_{\text{B}}T$ for ideal gases and $P \propto \rho^{5/3}$ for metals.

If there is no time-dependence then "Einstein's cosmological constant" Λ has $w = -1 =$ constant. Recall the Einstein equation is

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{where} \quad T^{\mu}{}_{\nu} = \text{diag}\left(-\rho c^2, P, P, P\right) \quad (\text{“Einstein equation”})\,.} \tag{3}$$

The LHS is some kind of combination of 1st and 2nd derivatives of the metric. Einstein realized you could make a static universe if you add a 'constant' $\Lambda g_{\mu\nu}$, which is a perfectly acceptable solution because energy is still conserved, i.e. $\nabla_\gamma(\Lambda g_{\mu\nu}) = 0$. However, when Hubble discovered the expanding universe Einstein abandoned his cosmological constant and deemed it his "greatest blunder". It was only recently reintroduced in order to explain cosmic acceleration, which could have been one of his greatest predictions!

Q: What are the units of Λ?
A: We first need to know the dimensions of the metric. Think about what happens in SR:

$$\left[ds^2\right] = \text{cm}^2 \qquad \Rightarrow \qquad [\eta_{\mu\nu} dx^\mu dx^\nu] = [\eta_{\mu\nu}]\,\text{cm}^2 = \text{cm}^2 \qquad \Rightarrow \qquad [\eta_{\mu\nu}] = \text{unitless}\,.$$

$$\text{Therefore, if } [R_{\mu\nu}] = \left[\partial^2/\partial x^2 g_{\mu\nu}\right] = \text{cm}^{-2} \text{ then } [\Lambda] = \text{cm}^{-2}\,.$$

General relativity has the simplifying feature that the effects of gravity can be transformed away in locally freely-falling reference frames. That is we can always choose a reference frame that locally reduces to special relativity. In flat space the curvature terms vanish, i.e. $R_{\mu\nu} = 0$, $R = 0$, and $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$. The field equation becomes

$$\Lambda\eta_{\mu\nu} \simeq \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \Leftrightarrow \qquad \begin{pmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix} = \frac{8\pi G}{c^4}\begin{pmatrix} -\rho_{\rm vac}c^2 & 0 & 0 & 0 \\ 0 & P_{\rm vac} & 0 & 0 \\ 0 & 0 & P_{\rm vac} & 0 \\ 0 & 0 & 0 & P_{\rm vac} \end{pmatrix}.$$

Finally, this relates the cosmological constant Λ to the vacuum density $\rho_{\rm vac}$ and pressure $P_{\rm vac}$ by

$$\boxed{\Lambda \equiv \frac{8\pi G}{c^2}\rho_{\rm vac}} \quad \text{or} \quad \boxed{\rho_{\rm vac} \equiv \frac{\Lambda c^2}{8\pi G}} \quad \text{and} \quad \boxed{P_{\rm vac} = -\rho_{\rm vac}c^2} \qquad (\text{“Cosmological constant”}). \tag{4}$$

All of this was done self-consistently. For future reference the present-day normalized density is

$$\Omega_\Lambda \equiv \frac{\rho_{\rm vac}}{\rho_{\rm crit,0}} = \frac{\Lambda c^2}{3H_0^2} \qquad \Rightarrow \qquad \Lambda = \frac{3\Omega_\Lambda}{R_{\rm H}^2}.$$

Dynamics

The Einstein equation for cosmic dynamics is exactly the same as the Newtonian version but has an effective density $\rho_{\rm eff} = \rho + 3P/c^2$:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \qquad (\text{“Cosmic dynamics”}).} \tag{5}$$

Recall: $F = m\ddot{r} = -GmM/r^2 \quad \Rightarrow \quad \ddot{r}/r = -GM/r^3 = -4\pi G\rho/3$ but $\rho \to \rho_{\rm eff} = \rho + 3P/c^2$. Using the equation of state for vacuum energy, i.e. $P_{\rm vac} = w\rho_{\rm vac}c^2$, yields

$$\rho_{\rm eff,vac} = \rho_{\rm vac} + \frac{3P}{c^2} = \rho_{\rm vac}(1 + 3w).$$

(Positive) acceleration requires $1 + 3w < 0$ or $\boxed{w < -\frac{1}{3}}$ so a cosmological constant ($w = -1$) satisfies this requirement.

The dynamics of a Λ dominated universe is determined by the Friedmann equation:

$$\begin{aligned} & \frac{\dot{a}}{a} = H_0\sqrt{\Omega_\Lambda} = c\sqrt{\frac{\Lambda}{3}} = \text{constant} \\ \Rightarrow \quad & \int_1^a \frac{da'}{a'} = \sqrt{\frac{\Lambda}{3}}\int_{t_0}^t c\,dt' \\ \Rightarrow \quad & a = \exp\left[\sqrt{\frac{\Lambda}{3}}(ct - ct_0)\right] \qquad (\text{“Exponential growth”}). \end{aligned}$$