AST 353 Astrophysics — Lecture Notes

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GENERAL RELATIVITY

The relativistic field equation

Last time we were well on our way to establishing the relativistic version of the Poisson equation:

$$\boxed{\nabla^2\varphi = 4\pi G\rho\,.} \tag{1}$$

The right hand side was generalized by using the stress energy tensor

$$\boxed{T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \text{(“Stress-energy tensor”).}} \tag{2}$$

Recall that a tensor is a sort of generalized vector so it is independent of the coordinate system used. Formally this means a tensor transforms in a specific manner under a change of coordinates from $x^\mu = (x^0, x^1, x^2, x^3)$ to $x^{\mu'} = (x^{0'}, x^{1'}, x^{2'}, x^{3'})$

$$p^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} p^\mu$$

for a vector and the stress energy tensor

$$T_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T_{\mu\nu}$$

provides an example for higher rank tensors.

In seeking a tensorial field equation we made the guess

$$-\nabla^2 g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,,$$

but this cannot possibly be what we are after because ∇^2 is not a tensor operation. It is not invariant! Another reason is that ∇^2 only acts on spatial coordinates. In relativity space and time are treated on equal footing. We can try to incorporate time by using the d'Alembertian operator

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,,$$

which is $-\partial^2/\partial(ct)^2 + \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in cartesian coordinates. Furthermore, conservation of energy places a constraint on the form of $T_{\mu\nu}$. In analogy to electromagnetism (i.e. $\vec{\nabla} \cdot \vec{B} = 0$) we require that the stress energy tensor is divergence-less under a generalized Div operator. To summarize, our requirements for the LHS, which we will call the Einstein tensor $G_{\mu\nu}$ are:

(i) Tensorial second derivatives in curved spacetime and

(ii) Energy conservation requires that the tensor satisfy $\mathrm{Div}(G_{\mu\nu}) = 0$.

The only combination obeying these properties is the Einstein tensor(!):

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \text{(Relativistic field equation).}} \tag{3}$$

Here the Ricci tensor is built up from the Christofel symbols:

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \qquad \text{(Ricci tensor),} \tag{4}$$

and the Ricci scalar R is the contraction of the Ricci tensor

$$R = g^{\mu\nu}R_{\mu\nu} \qquad \text{(Ricci scalar).} \tag{5}$$

The Einstein field equation represents 10 independent equations as $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric.

Total source of gravity

The "effective density" ρ_{eff} is taken as the contraction (or "trace") of the stress energy tensor

$$\boxed{\rho_{\text{eff}} = \frac{1}{c^2}\left(T_{00} + T_{11} + T_{22} + T_{33}\right) = \rho + \frac{3P}{c^2}\,.} \tag{6}$$

This additional source of gravity from pressure contributes to the overall density and makes possible the gravitational collapse of massive stars to black holes. This also shows why gravity in GR is more complicated. As mass is added it couples with other gravitational contributions, making the equations nonlinear and very complicated! Schematically Eq. 3 can be understood as

$$\begin{pmatrix}\text{curvature}\\ \text{of spacetime}\end{pmatrix} = \frac{8\pi G}{c^4}\begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix}\,. \tag{7}$$

Eq. 7 relates the fact that matter tells space how to curve, and space tells matter how to move. **Note:** Einstein's "greatest blunder" was to also add a cosmological constant $\Lambda g_{\mu\nu}$ to the left hand side of Eq. 3. Although he abandoned it with the discovery of Hubble expansion the term is now standard for the mysterious 'dark' energy component.

Hydrostatic equilibrium in GR

To apply GR to stellar equations we need to generalize the equation of hydrostatic equilibrium:

$$\frac{P}{r} = -\rho g = -\rho\frac{Gm(r)}{r^2} \qquad \text{(Newtonian HSE).} \tag{8}$$

The following alterations must be made for our pseudo-Newtonian argument:

(i) Here ρ is the "measure of inertia" so we account for pressure contributions. Replace:

$$\rho \to \rho + \frac{P}{c^2} = \rho\left(1 + \frac{P}{\rho c^2}\right)\,.$$

(ii) Mass m is a source of gravity so it is an effective mass. Replace:

$$m \to m_{\text{eff}} = \frac{4\pi r^3}{3}\rho_{\text{eff}} = \frac{4\pi r^3}{3}\rho + 4\pi r^3\frac{P}{c^2} = m + 4\pi r^3\frac{P}{c^2} = m\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\,.$$

(iii) Space in the gravitational field is revised. Replace:

$$r^2 \to r^2 \left(1 - \frac{2Gm}{rc^2}\right) .$$

The result is a direct replacement in the Newtonian equation of HSE:

$$\frac{dP}{dr} = -\rho \left(1 + \frac{P}{\rho c^2}\right) \frac{Gm \left(1 + \frac{4\pi r^3 P}{mc^2}\right)}{r^2 \left(1 - \frac{2Gm}{rc^2}\right)} ,$$

which is expressed as small deviations from the original version

$$\boxed{\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \qquad \text{(Oppenheimer–Volkoff).}} \qquad (9)$$

Note: We recover the Newtonian expression for weak fields $\varphi \sim \frac{Gm}{rc^2} \to 0$ and low pressure $P \ll \rho c^2$. Also if $c \to \infty$ we recover Newton's action at a distance.