AST 353 Astrophysics — Lecture Notes

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GENERAL RELATIVITY

Motion of particles

The equation of motion (e.o.m) for particles is determined by the Euler-Lagrange formalism:

$$\boxed{\int_A^B ds = \text{Extremal} \qquad \Longleftrightarrow \qquad \delta \int_A^B ds = 0 \qquad \Longleftrightarrow \qquad \delta \int_A^B L d\tau = 0\,.} \tag{1}$$

Here the simplified metric is $g_{\alpha\beta} = \text{diag}(g_{00}, g_{11})$ so the spacetime interval is

$$\boxed{ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{00}\left(dx^0\right)^2 + g_{11}\left(dx^1\right)^2\,,} \tag{2}$$

and the Lagrangian is

$$\boxed{L \equiv \frac{ds}{d\tau} = \sqrt{g_{00}\left(\dot{x}^0\right)^2 + g_{11}\left(\dot{x}^1\right)^2}\,.} \tag{3}$$

Therefore, the Euler-Lagrange equation is written for both indices $\alpha \in \{0, 1\}$

$$\boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}\,.} \tag{4}$$

We first work out the time equation:

$$\begin{aligned}
\text{RHS:} \quad & \frac{\partial L}{\partial x^0} = \frac{1}{2L}\left[\frac{\partial g_{00}}{\partial x^0}\left(\dot{x}^0\right)^2 + \frac{\partial g_{11}}{\partial x^0}\left(\dot{x}^1\right)^2\right] \\
\text{LHS:} \quad & \frac{\partial L}{\partial \dot{x}^0} = \frac{1}{2L}\left[2 g_{00}\dot{x}^0\right] \\
\Rightarrow \quad & \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^0}\right) = \frac{1}{L}\left[\frac{\partial g_{00}}{\partial x^0}\dot{x}^0\dot{x}^0 + \frac{\partial g_{00}}{\partial x^1}\dot{x}^1\dot{x}^0 + g_{00}\ddot{x}^0\right] \quad \text{where} \quad \ddot{x}^0 = \frac{d^2x^0}{d\tau^2}\,.
\end{aligned}$$

Putting these together gives:

$$\ddot{x}^0 = -\frac{1}{2}\frac{1}{g_{00}}\left[\frac{\partial g_{00}}{\partial x^0}\dot{x}^0\dot{x}^0 + \frac{\partial g_{00}}{\partial x^1}\dot{x}^0\dot{x}^1 + \frac{\partial g_{00}}{\partial x^1}\dot{x}^1\dot{x}^0 - \frac{\partial g_{11}}{\partial x^0}\dot{x}^1\dot{x}^1\right]\,.$$

Here we will introduce the inverse metric $g^{\alpha\beta} = \text{diag}(\frac{1}{g_{00}}, \frac{1}{g_{11}})$ which satisfies matrix multiplication to get the identity, i.e. $g_{\alpha\gamma}g^{\gamma\beta} = \delta_\alpha^\beta$ where δ_α^β is the Kronecker delta symbol equal to 1 if $\alpha = \beta$ and 0 otherwise. Thus, the calculation simplifies to

$$\ddot{x}^0 + \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^0}\dot{x}^0\dot{x}^0 + \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^1}\dot{x}^0\dot{x}^1 + \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^1}\dot{x}^1\dot{x}^0 - \frac{1}{2}g^{00}\frac{\partial g_{11}}{\partial x^0}\dot{x}^1\dot{x}^1 = 0$$

or

$$\ddot{x}^0 + \Gamma^0_{00}\dot{x}^0\dot{x}^0 + \Gamma^0_{01}\dot{x}^0\dot{x}^1 + \Gamma^0_{10}\dot{x}^1\dot{x}^0 + \Gamma^0_{11}\dot{x}^1\dot{x}^1 = 0\,.$$

The new $\Gamma^{\alpha}_{\beta\gamma}$ are called Christoffel symbols! Their form is 'simplified' because we are in a 2D diagonal spacetime:

$$\Rightarrow \qquad \Gamma^0_{00} = \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^0} \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^1} \qquad \Gamma^0_{11} = -\frac{1}{2}g^{00}\frac{\partial g_{11}}{\partial x^0}\,.$$

From this observation we get one of the famous geodesic equation:

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \text{(Geodesic equation)},} \tag{5}$$

where in a general spacetime (4D or indices from 0 to 3) we have

$$\boxed{\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right) \qquad \text{(Christoffel symbols).}} \tag{6}$$

Note: Equation 5 can also be written $\dot{u}^\alpha = -\Gamma^\alpha_{\mu\nu}u^\mu u^\nu$ where $u^\alpha \equiv dx^\alpha/d\tau$ and $\dot{u}^\alpha \equiv d^2x^\alpha/d\tau^2$. Also if we adopt the notation of using commas to denote derivatives ($f_{,\alpha} \equiv df/dx^\alpha$) then Eq. 6 simplifies to an expression that is much easier to remember: $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.

These equations are general, however, they only account for gravity. To allow for other forces (e.g. electromagnetism) we make an analogy to SR where the Christoffels vanish (i.e. $\Gamma^\alpha_{\mu\nu} \to 0$):

$$\boxed{m_0\ddot{x}^\alpha = f^\alpha \quad \text{(Newton/SR)}} \qquad \to \qquad \boxed{m_0\left[\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right] = f^\alpha \quad \text{(Einstein/GR).}} \tag{7}$$

Again, the $\Gamma^\alpha_{\mu\nu}$ term gives the "force-like" correction from gravitational curvature and the f^α term represents all nongravitational forces. We usually take $f^\alpha = 0$ for black holes and neutron stars because we are only considering gravity.

Note: In SR the metric coefficients are constant so the Christoffels vanish. Thus, $\Gamma^\alpha_{\mu\nu} = 0$ and $\ddot{x}^\alpha = 0$ in their inertial frame of reference. In GR we can locally consider a freely-falling reference frame near event A. Thus, we will still often use the equation

$$\boxed{\left.\frac{d^2x^\alpha}{d\tau^2}\right|_A = 0\,.}$$

End "Off the record" section

Motion of photons

Consider photons moving in SR with $dy = dz = 0$. The universality of the speed of light requires

$$c = \frac{dx}{dt} \qquad \Rightarrow \qquad ds^2 = -c^2dt^2 + dx^2 = 0\,,$$

so that "photons travel along null-geodesics." This is also true in GR because of our freedom to choose a locally freely-falling frame ($ds* = 0$).

The relativistic field equation

Recall the Newtonian field equation:

$$\boxed{\nabla^2\varphi = 4\pi G\rho\,.} \tag{8}$$

The Einstein field equation cannot be derived! However, we can arrive at what they should be if we consider the following requirements: *(i)* GR needs to reduce to Newton's laws in the applicable regimes and *(ii)* GR needs to pass physical tests. Einstein's gravity does exactly this!

Now recall the spacetime interval for weak gravity, i.e. the geometric version of Newton's laws:

$$\boxed{ds^2 = -c^2\left(1+\frac{2\varphi}{c^2}\right)dt^2 + dx^2 + dy^2 + dz^2\,.} \tag{9}$$

The first term is $g_{00} = -(1+2\varphi/c^2)$, so we roughly expect

$$-\nabla^2 g_{00} = \frac{2}{c^2}\nabla^2\varphi = \frac{8\pi G}{c^2}\rho\,.$$

However, in GR we also consider additional sources of gravity:

Newton: ρ – mass density only

Einstein: ρc^2 – mass-energy density and

P – pressure [erg cm^{-3}] with contributions in all 3 spatial directions

We can neatly organize these contributions by introducing the stress-energy tensor $T_{\mu\nu}$. In the fluid rest frame there are no off-diagonal components:

$$\boxed{T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \text{(“Stress-energy tensor”).}} \tag{10}$$

Considering $T_{00} = \rho c^2$ we have found the RHS of the field equation (the LHS is still not correct):

$$-\nabla^2 g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}\,.$$