AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith
(Dated: March 21, 2013)

GENERAL RELATIVITY

Index-gymnastics

In GR we use the 'Einstein summation convention' so we do not write the summands $\sum$,

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \equiv \sum_\alpha \sum_\beta \eta_{\alpha\beta} dx^\alpha dx^\beta \,.$$

In 3-dimensions we have 16 terms to expand (!) but if we restrict to one time dimension and one spatial dimension then we can explicitly illustrate how the summation is done:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{00} dx^0 dx^0 + \eta_{01} dx^0 dx^1 + \eta_{10} dx^1 dx^0 + \eta_{11} dx^1 dx^1 \,.$$

The road toward GR

We now reformulate Newton's theory in terms of spacetime language. The gravitational redshift also tells us about the flow of time if we consider 'light-clocks' with $\nu = c/\lambda = 1/\Delta t$. Therefore,

$$\nu_\infty = \nu_{\text{em}} \left(1 - \frac{GM}{rc^2}\right) \quad \Rightarrow \quad \frac{1}{\Delta t_\infty} = \frac{1}{\Delta t_{\text{em}}} \left(1 - \frac{GM}{rc^2}\right) \quad \Rightarrow \quad \Delta t_{\text{em}} = \Delta t_\infty \left(1 - \frac{GM}{rc^2}\right) \,.$$

Proper time is the time measured by a clock in the rest frame at r so the change in emitted time is simply $\Delta t_{\text{em}} = \Delta\tau$. The time measured at infinity is the "coordinate time" (i.e. $\Delta t_\infty = \Delta t$), an arbitrary but good choice for the baseline for time because there is 'no gravity' as $r \to \infty$.

$$\boxed{\Delta\tau = \left(1 - \frac{GM}{rc^2}\right) \Delta t \,.} \tag{1}$$

The connection to the spacetime interval is immediate:

$$ds^2 = -c^2 d\tau^2 = -c^2 \left(1 - \frac{GM}{rc^2}\right)^2 dt^2 \,.$$

However, in this frame the clock is at rest (i.e. $dx = dy = dz = 0$) so to good approximation we can put these elements back in the spacetime interval. Additionally, we identify the weak-field Newtonian potential so $-GM/r \to \varphi$ and Taylor expand to lowest order in $1/c^2$, i.e. $(1+\varphi/c^2)^2 \approx (1 + 2\varphi/c^2)$. The geometry of Newtonian gravity is contained in the spacetime interval

$$\boxed{ds^2 = -c^2 \left(1 + \frac{2\varphi}{c^2}\right) dt^2 + dx^2 + dy^2 + dz^2 \,.} \tag{2}$$

The goal now is to use this formulation of "Newtonian geometry" to proceed to a general geometry. The reason this is a geometric description is because the gravitational potential modifies distances. Thus, all the information is contained in the metric $g_{\alpha\beta}$. The coefficients are the key!

$$\text{In general,} \quad \boxed{ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta} \qquad \text{where} \qquad g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \,. \tag{3}$$

The metric tensor is symmetric upon exchange of indices so that $g_{\alpha\beta} = g_{\beta\alpha}$ which reduces the number of independent coefficients from sixteen to ten. We use the tensor formalism in analogy to the Newtonian field equation because it is independent of the coordinate system. We summarize the change from Newtonian to Einsteinian gravity:

$$\varphi \ : \ \text{scalar, 1 equation} \qquad \longrightarrow \qquad g_{\alpha\beta} \ : \ \text{metric tensor coefficients, 10 equations.}$$

Motion of particles

Rule: Freely falling particles travel on **geodesics** or 'straightest' paths through spacetime.
To find this "world line" we use the Calculus of Variations to minimize the spacetime interval:

$$\boxed{\int_A^B ds = \text{Extremal} \qquad \Longleftrightarrow \qquad \delta \int_A^B ds = 0\,.} \tag{4}$$

Begin "Off the record" section

In general the 'world line' events is described as a function of proper time, i.e. $x^\alpha = x^\alpha(\tau)$. Most of the physics and intuition can be gained from considering only one time coordinate and one spatial coordinate. Thus, our coordinate system and metric are respectively

$$\boxed{x^\alpha(\tau) = \big(x^0(\tau), x^1(\tau)\big) = \big(ct(\tau), x(\tau)\big)} \qquad \text{and} \qquad \boxed{g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}\,.} \tag{5}$$

With this choice the general line element is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{10} dx^1 dx^0 + g_{11} dx^1 dx^1\,.$$

The metric tensor is symmetric so $g_{01} = g_{10}$. However, we want this as simple as possible so we get rid of them entirely by choosing 'orthogonal coordinates' which we are free to do! Thus, $g_{01} = g_{10} = 0$ and the spacetime interval simplifies to

$$\boxed{ds^2 = g_{00}\left(dx^0\right)^2 + g_{11}\left(dx^1\right)^2\,.} \tag{6}$$

Finally, we arrive at the Lagrangian by taking the derivative with respect to proper time:

$$\int ds = \int \frac{ds}{d\tau} d\tau = \int L d\tau\,.$$

The Lagrangian, with proper velocities defined by overdots ($\dot{x}^0 \equiv dx^0/d\tau$ and $\dot{x}^1 \equiv dx^1/d\tau$) is

$$\boxed{L \equiv \frac{ds}{d\tau} = \sqrt{g_{00}\left(\dot{x}^0\right)^2 + g_{11}\left(\dot{x}^1\right)^2}\,.} \tag{7}$$

The action is minimized,

$$\delta \int_A^B L\big(x^0, \dot{x}^0, x^1, \dot{x}^1\big) d\tau = 0\,,$$

and the Euler-Lagrange equation is written for both indices $\alpha \in \{0, 1\}$

$$\boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}\,.} \tag{8}$$