AST 353 Astrophysics — Lecture Notes

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GENERAL RELATIVITY

Review

The big idea of general relativity (GR) is that "gravity is the curvature of spacetime." Einstein came up with the conceptual ideas before he figured out the math. Because of this we can actually understand quite a bit of the theory without getting bogged down by the abstract details. Ideas come in the form of thought experiments. For example, the equivalence principle equates gravity to acceleration, but there is a way to distinguish the two – Gravity induces tidal forces on freely falling objects. Thus, gravity acts as a lens to focus (free-fall) geodesics. The connection is that this happens in curved spaces as well so geometry is gravity!

The next thing we need is a field version of Newtonian gravity which will be used to check the validity of our theory (e.g. GR). The equations we derived are:

$$\boxed{\nabla^2\phi = 4\pi G\rho \qquad \text{(Poisson's Equation)}} \tag{1}$$

and

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\vec{\nabla}\phi \qquad \text{(Newtonian Equation of Motion).}} \tag{2}$$

Also recall the "pseudo-Newtonian" arguments for gravitational redshift:

$$\boxed{\nu_\infty = \nu_{\text{em}}\left(1 + \frac{\phi}{c^2}\right) \qquad \text{(Pseudo-Newtonian Gravitational Redshift),}} \tag{3}$$

which can be expressed with a dimensionless doppler shift

$$\frac{\Delta\nu}{\nu_{\text{em}}} = \frac{\nu_\infty - \nu_{\text{em}}}{\nu_{\text{em}}} = \frac{GM}{Rc^2}\,.$$

This is typically small and from the quiz we learned $GM_\odot/R_\odot c^2 \sim 10^{-6}$.

Special Relativity (SR)

We have already covered the basic concepts but to be sure we will discuss some formalism and results. Events in inertial systems (IS) can be described in different coordinate systems by a generalized vector:

$$A = (t, x, y, z) = (t', x', y', z')\,.$$

However, because time is fundamentally different than space we can no longer apply a simple Galilean coordinate transformation (e.g. $x' = x - vt$). Instead we use a Lorentz transformation:

$$\Delta x' = \frac{\Delta x - c\Delta t}{\sqrt{1-(v/c)^2}}\,, \qquad \Delta y' = \Delta y\,, \qquad \Delta z' = \Delta z\,, \quad \text{and} \qquad \Delta t' = \frac{\Delta t - (v/c^2)\Delta x}{\sqrt{1-(v/c)^2}}\,. \tag{4}$$

The differences can be made arbitrarily small so that $\Delta x \to dx$. Thus, the spacetime interval corresponding to this Lorentz transformation is

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2\,.} \tag{5}$$

Now the **proper time** is always measured in the rest-frame where the spatial coordinates do not change (i.e. $d\tau = dt'$ and $dx' = dy' = dz' = 0$). In this frame the spacetime interval is simplified:

$$\boxed{ds^2 = -c^2d\tau^2\,.} \tag{6}$$

We can write the spacetime interval in a neat (compact) way by viewing it as a matrix multiplication. If we represent the **Minkowski metric** $\eta_{\mu\nu}$ by the diagonal matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,$$

and a general differential distance as the vector $dx^\mu = (cdt, dx, dy, dz)^T$ where ^{T} denotes transpose then the spacetime interval can be written as

$$ds^2 = \sum_\mu \sum_\nu \eta_{\mu\nu} dx^\mu dx^\nu = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} = -c^2dt^2 + dx^2 + dy^2 + dz^2\,.$$

An even neater way is to use the "Einstein summation convention" where we sum over repeated indices which will be important in GR so get used to the following(!):

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad \text{(Minkowski Line Element).}} \tag{7}$$

In SR spacetime is said to be "flat" so what does this mean? For our purposes we say the metric can be transformed so that the derivatives of the metric vanish (i.e. $\partial\eta_{\mu\nu}/\partial x^\alpha = 0$). Note however that different coordinate systems could be more complicated. For example, the Minkowski metric in spherical coordinates is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, r, r\sin\theta)$. In GR we divorce the coordinate system from the physics! True physics originates from proper time, not coordinate time.

We conclude the lecture by writing the (SR) laws of motion in a coordinate-invariant way. We have already seen this because the spacetime interval is a Lorentz invariant quantity. There are others! For example, velocity is defined according to the proper time as

$$v^\alpha = \frac{dx^\alpha}{d\tau} = \gamma\frac{dx^\alpha}{dt} = \gamma\left(c, \vec{v}\right)\,, \tag{8}$$

where $\vec{v}$ is the 3-dimensional coordinate velocity we are used to. Likewise the "4-momentum" is

$$p^\alpha = m_0 v^\alpha = \left(\frac{\epsilon}{c}, \vec{p}\right)\,, \tag{9}$$

where m_0 is the rest mass, ϵ is the energy, and $\vec{p}$ is the familiar 3-dimensional momentum. Now we have a method to write the laws of nature in an invariant form:

If Newton says

$$\vec{F} = m_0\vec{a}\,,$$

then Einstein says

$$f^\alpha = \frac{dp^\alpha}{d\tau}\,.$$

Force-free motion requires $dp^\alpha/d\tau = 0$.

Note: The equivalent of an 'inertial' frame in SR is a 'freely-falling' frame in GR.