AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith*

(Dated: March 5, 2013)

INTRODUCTION TO GENERAL RELATIVITY

Review for the Exam

Memorize the following equations for OoMA approximations:

- Free-fall time: $\tau_{\text{ff}} \approx \frac{1}{\sqrt{G\rho}}$
- Virial Theorem: $E_{\text{kin}} \approx |E_{\text{pot}}|$ where $|E_{\text{pot}}| \approx \frac{GM^2}{R}$ and $E_{\text{kin}} \approx N k_B T$
- Central pressure: $P_c \approx \frac{GM^2}{R^4}$

Remember to bring a calculator and study the previous homework and quizzes.

Introduction to General Relativity

Einstein's Theory of General Relativity is an incredibly elegant and seductive theory of gravity. This is bonus portion of the class. We will present the details as we need them. For starters we will remind ourselves of what motivated Einstein toward the theory.

Newton knew his "action at a distance" was an insane concept! How does a distribution of mass instantaneously know about they dynamics of the rest of the universe? Newton cannot explain the mechanism for gravity, just a set of equations to describe it. (Many would argue this is still the case for Einstein's equations!) However, Einstein says gravity is acceleration, which is a clear concept in our minds. The equivalence principle makes gravity intuitive because acceleration can be accounted for by changing coordinate systems. The 'special frame' is the one in free-fall!

Gravity tends to focus free-fall paths! This can be understood as an artificial 'tidal force' or by converging 'geodesics' on curved surfaces. Einstein realized that he did not know how to model this concept though so he says to his friend Marcel Grossman "I need help with the geometry, otherwise I go mad with this stuff!"

General Relativity (GR) Formalism

Recall the universal law of gravitation for two particles:

$$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r \quad \text{or} \quad \vec{F} = m\vec{g}\,.$$

However, the mass here is the 'gravitational mass' m_g because it measures how strongly the object is coupled to the gravitational field.

*asmith@astro.as.utexas.edu

We may introduce this in terms of a gravitational potential $\phi = -GM/r$ for a point source:

$$\boxed{\vec{g} = -\vec{\nabla}\phi\,.} \tag{1}$$

Or re-phrase it by taking the divergence and identifying the Laplacian operator ($\nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla}$) so

$$\vec{\nabla}\cdot\vec{g} = -\nabla^2\phi\,.$$

If we integrate over a surface specified by $d\vec{A}$ and apply Gauss's Theorem then

$$-\int \nabla^2\phi dV = \int \vec{\nabla}\cdot\vec{g}dV = \int \vec{g}\cdot d\vec{A} = -4\pi r^2 g = -4\pi G\int \rho(r)dV$$

and we get Poisson's equation:

$$\boxed{\nabla^2\phi = 4\pi G\rho \qquad \text{(Poisson's Equation).}} \tag{2}$$

This is the "field equation for Newtonian gravity" and gives an equation for the field for all $\vec{r}$.

The Equation of motion is given by applying Newton's Second Law, where the mass for this law is the 'inertial mass' m_i,

$$m_i\frac{d^2\vec{r}}{dt^2} = \vec{F} = m_g\vec{g}\,,$$

and it is not at all obvious that $m_i = m_g = m$! We shall take this as an experimental fact as shown by Galileo at the tower in Pisa. (We shall see that Einstein's theory predicts the equivalence of the different types of masses.) The next step is to cancel the masses to get the equation of motion

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\vec{\nabla}\phi \qquad \text{(Equation of Motion).}} \tag{3}$$

Gravitational Redshift

We will now use a pseudo-Newtonian argument to explain the gravitational redshift of light moving away from a massive object such as a planet or star. Originally the emitted energy of the photon at the surface R is $\epsilon_{\rm em} = h\nu_{\rm em}$ but far away from the star we find a different energy given by $\epsilon_\infty = h\nu_\infty$. The photon losses energy, which comes from the work required to climb out of the potential well of the star. If we define the "mass" of the photon as

$$m_{\rm ph} \equiv \frac{\epsilon_{\rm em}}{c^2}$$

then the work done is

$$W = -\int_R^\infty \vec{F}\cdot d\vec{r} = \frac{GMm_{\rm ph}}{R} = \frac{GM\epsilon_{\rm ph}}{Rc^2}\,.$$

Therefore, the energy has been lowered by a small amount:

$$\epsilon_\infty = \epsilon_{\rm em} - \frac{GM\epsilon_{\rm ph}}{Rc^2} = \epsilon_{\rm em}\left(1 - \frac{GM}{Rc^2}\right)\,,$$

and the frequency has shifted by the same factor

$$\boxed{\nu_\infty = \nu_{\rm em}\left(1 + \frac{\phi}{c^2}\right) \qquad \text{(Pseudo-Newtonian Gravitational Redshift),}} \tag{4}$$

which is also approximately true in Einstein's GR. In fact, for weak gravitational fields it is off by a factor of two! This tells us something about time intervals (recall $\nu = \Delta N/\Delta t = 1/T$ where T is the period) and we can then think of light clocks in a similar manner as special relativity.