AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith*
(Dated: January 31, 2013)

CLASSICAL/QUANTUM GAS BOUNDARY

Review

Quiz: What is the temperature T of a (spherical) gas cloud with radius $R \sim 100$ pc ($\sim 3 \times 10^{20}$ cm) and mass $M \sim 10^6$ $\mathrm{M}_\odot$ ($\sim 2 \times 10^{39}$ g)? The OoMA answer is given by considering the pressure from two different directions. First we consider the energy and continue with the viral theorem:

$$P \approx u_{\rm kin} \approx |u_{\rm pot}| \approx \frac{GM^2}{RV} \, .$$

Next we pursue pressure from the direction of the ideal gas law:

$$P \approx n k_{\rm B} T \approx \frac{N}{V} k_{\rm B} T \approx \frac{M}{m_{\rm H} V} k_{\rm B} T \, .$$

Here we have assumed that this was all hydrogen so that the number of particles is given by the total mass M divided by the particle mass $m_{\rm H}$. Equating these two pressures gives

$$T \approx \frac{G M m_{\rm H}}{k_{\rm B} R} \approx 10^3 - 10^4 {\rm K} \, .$$

This is a cosmological "mini halo" where the real answer is around 5,000 K so we have done well!

Also we want to make sure this is reasonable in case we make a calculator or mental math mistake. The coldest astrophysical temperatures are limited by the pervasive cosmic microwave background (CMB) which is about 3 K. The hottest temperatures are probably those found as a supernova (SNe) explodes which is about 10^{10} K. Therefore a feeling for temperature situations is

$$1 \text{ K} \lesssim T \lesssim 10^{10} \text{ K} \quad \text{(Realistic Temperature Range)} \, .$$

Classical/Quantum Gas Boundary

Additional physics beyond the Classical + NR regime happens because the density of the star increases, perhaps because the main sources of thermal pressure (fusion) have been exhausted. When this happens we break the quantum barrier before the special relativistic limit.

So where does quantum pressure originate? The answer is the Pauli Exclusion Principle which prevents Fermions (electrons, protons, . . .) from being compressed to arbitrarily high densities.

When does Quantum Mechanics (QM) become important? The answer has to do with the particle/wave duality! QM is all about probabilities so in effect it smears out space on small scales. If two wave functions get too close to each other then the overlapping (Fermi) pressure pushes them apart. Let's return to the picture of N particles in a box of volume $V = L^3$. The average

*asmith@astro.as.utexas.edu

distance between particles ℓ is derived by considering that the volume around one particle is ℓ^3 and N of those volumes gives the total volume, i.e.

$$V_{\text{total}} = NV_{\text{one-particle}} \quad \Rightarrow \quad L^3 = N\ell^3 \quad \Rightarrow \quad \boxed{\ell = \left(\frac{V}{N}\right)^{1/3} \sim n^{-1/3}\,.} \tag{1}$$

Consider: The particle wavelength depends directly on Planck's constant $h = 6.626 \times 10^{-27}$ erg s and inversely on the particle momentum p so that QM effects kick in when

$$\boxed{\ell \lesssim \lambda_{\text{dB}} \equiv \frac{h}{p} \quad (\text{“de Broglie wavelength”})\,.} \tag{2}$$

Notes: Units of energy · time is called “action.” NR momentum is $p = m_0 v$.

Equation **??** means

$$n^{-1/3} \lesssim \frac{h}{p} \quad \Rightarrow \quad p \lesssim hn^{1/3}\,,$$

which happens with cold (slow-moving) dense gas is in danger of becoming degenerate. We can rephrase this in terms of temperature by considering

$$\frac{3}{2}k_{\text{B}}T = \epsilon_{\text{kin}} = \frac{p^2}{2m_0}\,.$$

However, this is an OoMA calculation so

$$k_{\text{B}}T \approx \frac{p^2}{m_0} \lesssim \frac{h^2 n^{2/3}}{m_0}\,.$$

In either case we get a temperature dependance of

$$\boxed{T \propto n^{2/3}\,.} \tag{3}$$

Quantum (“degenerate”) gas

Heisenberg Uncertainty: The topic of quantum distributions is advanced but all of the ‘simple’ ideas we discuss are true so you won't need to ‘unlearn’ them later. First we consider 6-dimensional phase space $(\vec{x}, \vec{p}) = (x, y, z, p_x, p_y, p_z)$ and we chop it up into small boxes of length Planck's constant h. Here we designate the spatial volume as V or equivalently V_x to have an unambiguous way to distinguish from the momentum volume V_p. The differential volume of these cells is $dV_x dV_p = d^3x\, d^3p = h^3$. This comes from the one dimensional Heisenberg Uncertainty Relation that $\Delta x \Delta p \gtrsim h$. (You may find other constants such as $\hbar = h/2\pi$ but we do not worry about such details.)

The Pauli Exclusion Principle: “Only 2 (or less) Fermions (e.g. electrons) are allowed per quantum cell” so we assume a tightly packed grid according to that rule! An ideal gas follows the same rules but is very sparse in the ‘grid’ we have built. The tight packing is called “complete degeneracy” and intermediate cases (between complete degeneracy and the ideal gas case) are called “partial degeneracy” which adds algebraic complexity but no new physical insight so we do not talk about them.

Particles are distributed into the lowest energy states first! Our model is to fill these levels completely and create a "surface of the Fermi sea" up to the Fermi momentum p_{F}. (As a preview we can now see why we eventually have to turn to special relativity, the Fermi energy ϵ_{F} becomes comparable to m_0c^2.) We can visualize this "sea" as an isotropic sphere in momentum space assembled one block at a time from the origin outward.

To get the total number of particles we use the (spin) degeneracy g_i multiplied by the total volume divided by the volume of one particle:

$$N = g_i \frac{V_x V_p}{V_{\mathrm{one-particle}}} = 2\frac{\frac{4\pi}{3} p_{\mathrm{F}}^3 V}{h^3} = \frac{8\pi}{3h^3} p_{\mathrm{F}}^3 V$$

but the number density is

$$n = \frac{N}{V} = \frac{8\pi}{3h^3} p_{\mathrm{F}}^3$$

so the final Fermi momentum is

$$\boxed{p_{\mathrm{F}} = h \left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}\,.} \tag{4}$$