AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith*
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PHYSICS OF COMPACT OBJECTS (CONTINUED)

Review

Last time we introduced the difference between 'local' and 'global' descriptions of stars. The equation of Hydrostatic Equilibrium (HSE)

$$\boxed{\frac{dP}{dr} = -\rho g\,,} \tag{1}$$

is an "exact" local law. The differential equation is valid for small regions, but it is valid at all regions. We also have a global law for the average pressure of a system, namely

$$\boxed{\langle P \rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V}\,.} \tag{2}$$

Sometimes we like global descriptions better because they reduce the number of quantities to describe the physics. Remember that we got this by looking at the microscopic physics (i.e. particles in a box) which means it is also very general!

Finally, we looked at two classes of particles, non-relativitstic (NR) and ultra-relativistic (UR). Our discoveries can be summarized as

$$\boxed{P = \frac{2}{3}u_{\rm kin} \quad (NR)} \qquad \text{and} \qquad \boxed{P = \frac{1}{3}u_{\rm kin} \quad (UR)\,.} \tag{3}$$

For NR particles we discovered a relation between the kinetic and potential energy known as the Virial Theorem:

$$\boxed{2E_{\rm kin} = -E_{\rm pot} \qquad \text{"Virial Theorem" (VT)}\,.} \tag{4}$$

This important result highlights a difference between NR and UR systems. Today we will find a relativistic version of the VT and show what this means for the stability of stars.

Total Energy and Stability (cont.):

From our previous experience with physics we know that the total energy $E_{\rm tot}$ comes from many sources. With the assumptions we have made so far we only have kinetic and potential contributions so that

$$E_{\rm tot} = E_{\rm kin} + E_{\rm pot}\,.$$

*asmith@astro.as.utexas.edu

A system is gravitationally bound (or stable) if

$$E_{\text{tot}} < 0 \qquad \text{(Stability Condition)}\,. \tag{5}$$

This is proven by considering the total energy at infinity. As $r \to \infty$ $E_{\text{pot}} \to 0$, but the kinetic energy is positive definite (i.e. $E_{\text{kin}} > 0$) so $E_{\text{tot},\infty} = E_{\text{kin},\infty} > 0$. This means that if a particle can escape then it has a non-negative energy and it would be a contradiction for lower energy bound particles to ever make it to infinity. Therefore, under our assumptions a bound system has negative total energy. **Note:** NR gas CAN be stable under Eq. 4.

The relativistic version of the Virial Theorem

We follow the same example as the previous (NR) VT section but this time with a UT gas. We need an average pressure of at least

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\text{pot}}}{V}$$

but what is available is

$$\langle P \rangle = \frac{1}{3}u_{\text{kin}} = \frac{1}{3}\frac{E_{\text{kin}}}{V}\,.$$

Therefore, for photons and UR particles

$$\boxed{E_{\text{kin}} = -E_{\text{pot}} \qquad \text{“Relativistic version of the Virial Theorem”}\,.} \tag{6}$$

The total energy is $E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = 0$ so the object is not stable (i.e. not ‘bound’) and small perturbations can lead to its destruction! Famous cases of this happening are supernova explosions.

Pressure inside stars

Now we want to talk about the real pressure inside stars, which is done by looking for the stellar equation of state (e.o.s). The ideas is that the pressure will be a function of known physical quantities such as density (n or ρ) and temperature T. Thus, $P = P(n, T, \ldots)$ where the ‘$\ldots$’ refers to things like chemical composition (the “composition of matter”) which is a mess and is best left to the experts.

We consider four different stellar regimes:

Classical (non-degenerate)	Quantum (degenerate)
Non-Relativistic (NR) $\epsilon_{\text{kin}} \ll m_0 c^2$ (slow)	Ultra-Relativistic (UR) $\epsilon_{\text{kin}} \gg m_0 c^2$ (fast)

Case 1: Classical + NR

This is the "normal" or "vanilla" type model and is the simplest case. From the equipartition theorem each (quadratic) degree of freedom has an average energy of $\frac{1}{2}k_{\rm B}T$. Here $k_{\rm B} = 1.38 \times 10^{-16}$ erg/K is the Boltzmann constant and is really just a change of 'physical currency' from energy to temperature. With three translational degrees of freedom the pressure for a NR gas is

$$P = \frac{2}{3}u_{\rm kin} = \frac{2}{3}n\langle\epsilon_{\rm kin}\rangle = \frac{2}{3}n\left(\frac{3}{2}k_{\rm B}T\right) = nk_{\rm B}T\,.$$

This is summarized as the important ideal gas law:

$$\boxed{P = nk_{\rm B}T \quad \text{(Ideal gas e.o.s.)}\,.} \tag{7}$$

Note: The number density is given by the number of particles divided by the volume, i.e. $n \sim \frac{N}{V}$.