AST 353 Astrophysics — Lecture Notes

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PHYSICS OF COMPACT OBJECTS (CONTINUED)

Review

Recall last time that we reviewed the equation of Hydrostatic Equilibrium (HSE)

$$\boxed{\frac{dP}{dr} = -\rho g\,.} \tag{1}$$

This is one of the most important equations in astrophysics because it quite accurately describes the condition of stable stars.

We also discussed timescales. Before you attempt an expensive computation you want to make sure you are getting the relevant information. For example, if the timescale of your interesting physics is 10^12 years then this is longer than the age of the Universe!

Finally remember that there are many ways to look at a problem, e.g.

$$P_c \approx \frac{GM^2}{R^4} = \frac{GM^2/R}{R^3} = \frac{\text{energy}}{\text{volume}}\,.$$

A different way to look at HSE:

We have already encountered two quantities which describe the strength of the gravity – $g = Gm/r^2$ and $\tau_{\text{ff}} \sim 1/\sqrt{G\rho}$. We gain intuition about what these numbers mean. Indeed, a short free-fall time means a stronger gravitational force ($\tau_{\text{ff},\odot} \sim 1$ hr whereas a white dwarf has $\tau_{\text{ff,WD}} \sim 1$ s). We now introduce the gravitational potential energy E_{pot} which is a measure of how expensive it is (in terms of work) to peel off all the shells of the star. By convention the potential at infinity in zero ($E_{\text{pot}}(r = \infty) \equiv 0$) and as it is a lower energy state to have the object bound by gravity we always think of negative potential energies ($E_{\text{pot}} < 0$). Thus, we define the potential energy in terms of an integral over all particles (or mass shells)

$$\boxed{E_{\text{pot}} = -\int_0^M \frac{Gm(r)}{r} dm\,.} \tag{2}$$

What average pressure $\langle P \rangle$ is needed to counteract gravity?

Recall from statistics that for a discrete sample with relative weights w_i for each value x_i the average of the ensemble is given by

$$\langle x \rangle \equiv \frac{\sum_i w_i x_i}{\sum_j w_j}\,.$$

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Thus for a continuous pressure distribution we average as follows:

$$\langle P\rangle = \frac{\int_0^R 4\pi r^2 P dr}{\int_0^R 4\pi r^2 dr} = \frac{1}{V}\int_0^R 4\pi r^2 P dr\,.$$

We make progress by performing integration by parts, i.e.

$$d(uv) = udv + vdu \Rightarrow \int udv = uv - \int vdu$$

$$\text{where in this case} \quad u = P(r) \qquad dv = 4pir^2 dr$$
$$\text{so that} \quad du = \frac{dP(r)}{dr}dr \qquad v = \frac{4\pi}{3}r^3\,,$$

which results in

$$\int_0^R 4\pi r^2 P dr = \frac{4\pi}{3}r^3 P(r)|_0^R - \frac{1}{3}\int_0^R 4\pi r^3 \frac{dP}{dr}dr\,.$$

Now, it is clear that $r^3 P|_{r=0} = 0$ and as before we can safely assume the pressure goes to zero at the surface of the star so that $r^3 P(r)|_{r=R} = R^3 P(R) = 0$. Therefore, we can use Eq. 1 for hydrostatic equilibrium to simplify what is under the integrand

$$4\pi r^3 \frac{dP}{dr}dr = 4\pi r^3\left(-\rho\frac{Gm}{r^2}\right)dr = -\frac{Gm}{r}\left(4\pi r^3 \rho dr\right) = -\frac{Gm}{r}dm\,. \tag{3}$$

Note: The mass $m = m(r)$ is a function of radius but we don't usually write it that way.

Putting the last few equations together gives the average pressure in terms of the gravitational potential energy,

$$\langle P\rangle = \frac{1}{V}\int_0^R 4\pi r^2 P dr = \frac{1}{3}\int_0^R \frac{Gm}{r}dm = -\frac{1}{3}\frac{E_{\rm pot}}{V}\,.$$

We emphasize this in boxed form

$$\boxed{\langle P\rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V}\,.} \tag{4}$$

.1. What is pressure?

We know that pressure exerts some kind of force but we want a way to understand this macroscopic concept in terms of the microphysics. To do this we use **kinetic theory**, which is very beautiful but can get quite complicated. It is helpful to think of pressure as "momentum flux". Dimensionally,

$$P = \frac{\Delta F}{\Delta A} = \frac{\Delta p}{\Delta t \Delta A}\,.$$

Particles in a box: Start by considering N particles in a box of sides L, area $A = L^2$, and volume $V = L^3$ so that the number density is $n = N/V$. Assume all of the particles are at the same temperature and have the same velocity v_x. We simplify the discussion by considering only one (x-) dimension. Thus the distance traveled in a small time Δt is $x = v_x \Delta t$ and the correspondingly small volume is $\Delta V = v_x \Delta t A$. However, only half of the particles move in a direction to hit the wall (the $+x$-direction) so the number of particles being transported is

$$\Delta N = \frac{1}{2} n \Delta V = \frac{1}{2} n v_x \Delta t A \,.$$

The total change in momentum Δp_x is given by considering collisions from all ΔN particles, each with momentum transfer $p_x - (-p_x) = 2p_x$. With this factor of two the pressure is given by

$$P = \frac{\Delta p_x}{\Delta t \Delta A} = \frac{2\Delta N p_x}{\Delta t \Delta A} = \frac{n A p_x v_x \Delta t}{\Delta t \Delta A} = n\langle p_x v_x \rangle \,.$$

(This is an average which follows a Maxwell-Boltzmann distribution.) However, we need to consider three dimensions which is actually quite straightforward if we assume an "isotropic pressure" so that the x, y, and z directions are all treated on equal footing. We simply get a factor of one third because $\langle p_x v_x \rangle = \frac{1}{3}(\langle p_x v_x \rangle + \langle p_x v_x \rangle + \langle p_x v_x \rangle) = \frac{1}{3}\langle \vec{p} \cdot \vec{v} \rangle$ and so that

$$\boxed{P = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle \,.} \tag{5}$$

Notice that this is very general. We can apply this to any system we like – gas, photons, or even galaxies where the particles are stars!

Pressure of an Ideal Gas:

The pressure of an ideal gas is given by the well-known formula

$$P = n k_B T \,,$$

where k_B is the Boltzmann constant and T is the absolute temperature of the gas. In special relativity the energy ϵ and the momentum p may be related to the rest mass m_0 via

$$\epsilon^2 - p^2 c^2 = m_0^2 c^4 \,.$$

[For multiple particles the total energy E is the sum of energies (i.e. $E = \sum \epsilon_i$), which may be as simple as $E = N\epsilon$.] We can safely assume particles are non-relativistic if their kinetic energy $\epsilon_{\text{kin}} = \epsilon - m_0 c^2$ is much less than their rest energy $m_0 c^2$, otherwise we assume a relativistic scenario. Finally, we introduce the kinetic energy density (as per volume) so that $u_{\text{kin}} \equiv E_{\text{kin}}/V$. We specialize to the two cases:

(i) Non-relativistic (NR) particles:

$$p = mv \quad \text{and} \quad \epsilon_{\text{kin}} = \frac{1}{2} m v^2 \quad \Rightarrow \quad pv = 2\epsilon_{\text{kin}} \,.$$

The pressure may therefore be calculated as

$$P = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle = \frac{2}{3}\frac{N\epsilon_{\text{kin}}}{V} = \frac{2}{3} u_{\text{kin}} \,.$$

We emphasize:

$$\boxed{P = \frac{2}{3} u_{\text{kin}} \quad (NR) \,.} \tag{6}$$

(ii) Ultra-relativistic (UR) particles:

$$v = c \quad \text{and} \quad \epsilon_{\rm kin} = pc \quad \Rightarrow \quad pv = \epsilon_{\rm kin}\,.$$

We then calculate and emphasize:

$$\boxed{P = \frac{1}{3} u_{\rm kin} \quad (UR)\,.} \tag{7}$$

The Virial Theorem (VT)

The factor of 2 difference between NR and UR particles is actually quite important! The reason comes out when we consider the stability of a system with the Virial Theorem. Consider a NR gas. We need an average pressure of at least

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V}$$

but what is available is

$$\langle P \rangle = \frac{2}{3}\frac{E_{\rm kin}}{V}\,.$$

Therefore, for a system to be in HSE we need

$$-\frac{1}{3}\frac{E_{\rm pot}}{V} = \frac{2}{3}\frac{E_{\rm kin}}{V}$$

which gives the desired theorem about the kinetic and potential energy in HSE:

$$\boxed{2E_{\rm kin} = -E_{\rm pot} \qquad \text{“Virial Theorem” (VT)}\,.} \tag{8}$$

Note: Astrophysicists love the virial theorem because we talk about large systems which we don't know much about. Therefore, average and total quantities are very useful.