AST 353 Astrophysics — Lecture Notes

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PHYSICS OF COMPACT OBJECTS

In this course we will mostly be dealing with dead stars (i.e. white dwarfs, neutron stars, and black holes). However, we must build up to these extreme cases. We will start with a simple but good model. We consider a "theorist's" idealized star:

- No magnetic fields
- No rotation
- Perfectly Spherical

Note: Real stars of course *do not* meet these criteria but to first order this is fairly good.

Units: Astronomers use the **CGS** (Centimeters–Grams–Seconds) system...NOT SI.
Thus, for the **sun** we have:

$$M_\odot \cong 2 \times 10^{33}\ \text{g}$$
$$R_\odot \cong 7 \times 10^{10}\ \text{cm}$$
$$\Rightarrow \quad \langle\rho\rangle_\odot = \frac{\text{Mass}}{\text{Volume}} \cong \frac{M}{\frac{4\pi}{3}R^3} \cong 1.4\ \text{g cm}^{-3}\,, \quad \text{which is about the density of water.}$$

This is important because a number by itself is not meaningful by itself, we need to compare it with something we know about to gain physical intuition. Hence, we use the sun.

Mechanical Structure

In mechanics the Equation of Motion (e.o.m.) naturally arises from Newton's 2^{nd} Law,

$$F_{\text{total}} = ma \qquad \text{or} \qquad \text{total force} = \text{mass} \times \text{acceleration}\,. \tag{1}$$

In order to determine what forces are involved we recall the assumptions (no rotation, etc.) and eliminate everything but *gravity* and a force from "pressure differences" as we move out from the center. (There is a huge pressure at the center and very little at the surface. We assume some smooth curve to connect the dots...)

Before we actually calculate these two forces we define a **mass coordinate**. The idea is to relate mass with radius because the spherical geometry gives us the same information either way. To do this we think of adding up very thin shells of mass dm and thickness dr at all radial values less than r. The mass interior to the final shell at r is called $m(r)$ and depends on the distance from the center r. The volume in a shell is $dV = 4\pi r^2 dr$ so the standard density relation gives

$$dm = 4\pi r^2 \rho dr\,,$$

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or upon integration

$$m(r) = 4\pi \int_0^r r'^2 \rho(r') dr' \,. \tag{2}$$

For a star with constant density $\rho = \rho_0$ this is trivial, however, this is not a realistic density profile!

The **gravitation force** on each shell is given by the gravitational acceleration $g(r)$ multiplied by the differential mass dm. Because of the spherical symmetry the gravitational acceleration is the same as if all the interior mass $m(r)$ was located at the center. Thus, the summary for gravity is:

$$F_{\text{grav}} = -g(r)dm = -\frac{Gm(r)}{r^2} dm \,.$$

The **pressure force** on each shell is found by considering the pressure difference across the shells. We call this the 'pressure gradient' because it involves the spatial derivative dP/dr. In fact, if we think of force as pressure times area then in a given volume $dV = (4\pi r^2)dr$ the forces for pressure combine as

$$\begin{aligned} F_{\text{pressure}} &= F_{\text{pressure}}(r) - F_{\text{pressure}}(r + dr) \\ &= 4\pi r^2 P(r) - 4\pi r^2 \left[P(r) + \frac{dP}{dr} dr \right] \\ &= -4\pi r^2 \frac{dP}{dr} dr \,. \end{aligned}$$

Putting these together we get the **total force**

$$F_{\text{total}} = -g dm - \frac{1}{\rho} \frac{dP}{dr} (4\pi r^2 \rho dr) = \left[-g dm - \frac{1}{\rho} \frac{dP}{dr} \right] dm \,.$$

But $F_{\text{total}} = \text{mass} \times \text{acceleration} = dm \cdot d^2r/dt^2$ so the mass dm falls out and the e.o.m. is

$$\boxed{\frac{d^2 r}{dt^2} = -g - \frac{1}{\rho} \frac{dP}{dr} \,.} \tag{3}$$

There are two things to take note of — First, $g = g(r)$ is a function of radius, and second, the pressure gradient is negative because it decreases in the outward direction.

Thought Experiment

What happens when we 'switch off' the pressure term? Remember the previous lecture's OoMA:

$$\begin{aligned} & \frac{d^2}{dt^2} = -g(r) \\ \Rightarrow \quad & -\frac{R}{\tau_{\text{ff}}^2} \approx -\frac{GM}{R^2} \\ \Rightarrow \quad & \boxed{\tau_{\text{ff}} \approx \frac{1}{\sqrt{GM/R^3}} \approx \frac{1}{\sqrt{G\rho}} \,.} \end{aligned} \tag{4}$$

Aside: What are the important time scales?

- Hubble-time (-scale) $\Rightarrow \tau_{\mathrm{H}} \cong 13.7$ Gyr — This is the age of the universe!
- Free fall time $\Rightarrow \tau_{\mathrm{ff}}$ — In order to gain intuition we compare this to $\tau_{\mathrm{ff},\odot} \sim 1$ hr. The sun hasn't changed for a long time ($\sim$Myr/Gyr) so 1 hr is an astronomical blink of the eye! This tells us that pressure balances gravity extraordinarily well!

Hydrostatic (Mechanical) Equilibrium

We have seen that on short timescales stars like our sun do not change very much. In this case, the time derivatives are zero (i.e. $d/dt = 0$) and Eq. 3 gives the equation of hydrostatic equilibrium:

$$\boxed{\frac{dP}{dr} = -\rho g\,.} \tag{5}$$

Note: This is a powerful result which should be memorized!

Simple Estimate for Central Pressure P_{c}

Now we desire to connect what we observe from the surfaces of stars (T, ρ, P,...) to the central or interior quantities. To do this we use OoMA on Eq. 5. We assume "zero boundary conditions" so that the pressure at the surface of the star goes to zero (i.e. $P_0 \to 0$). This is actually a fairly good approximation and makes life a lot easier!

$$\begin{aligned} \text{LHS:} \frac{dP}{dr} &\approx \frac{\Delta P}{\Delta r} = \frac{P_0 - P_{\mathrm{c}}}{R - 0} = -\frac{P_{\mathrm{c}}}{R} \\ \text{RHS:} -\rho g &\approx -\frac{M}{R^3}\frac{GM}{R^2} = -\frac{GM^2}{R^5}\,. \end{aligned}$$

Putting these together gives an estimate for the central pressure,

$$\boxed{P_{\mathrm{c}} \approx \frac{GM^2}{R^4}\,.} \tag{6}$$

We can use solar values to benchmark other solutions. Thus, $P_{\mathrm{c},\odot} \approx 10^{16}$ dyne cm$^{-2} \approx 10^{15}$ Pa, roughly ten orders of magnitude larger than the atmospheric pressure on earth.

Units: dyne cm$^{-2} \approx \frac{1}{10}$ Pa gives the CGS to SI conversion!
Also, remember the dimensional analysis of these energy and pressure:

$$\text{pressure} = \frac{\text{energy}}{\text{volume}} \sim [\mathrm{erg} \cdot \mathrm{cm}^{-3}]\,.$$