AST 353 Astrophysics — Lecture Notes

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ORDER-OF-MAGNITUDE ASTROPHYSICS

"Order of magnitude" or "back of the envelope calculations" often are quicker, require less effort, and provide more physical insight than a precise calculation.

Q: How long does it take a star to collapse to a single point if we consider only gravity?

The reason this collapses is because we are ignoring any kind of pressure. The assumptions are a sphere of radius R, mass M and constant uniform density ρ_0 . Naturally, the mass and density are related through the volume:

$$M = \frac{4}{3}\pi R^3 \rho_0$$
 at $t = 0$ and $M = \frac{4}{3}\pi r^3 \rho$ at later times.

We want to find the "feefall time" (or "dynamical time"). To do this we need an equation of motion (e.o.m.) which is found by combining Newton's 2nd Law, F = ma, with his universal law of gravitation, $F \propto 1/r^2$,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} = -\frac{4}{3}\pi G\rho r\,.$$
(1)

The last equality came from substituting ρ for M. There are two ways to solve this...

A1: Precise calculation

Borrow a trick from differential equations. Take the time derivative of the ' v^2 kinetic energy' and then use Eq. 1 to get

$$\frac{d}{dt}\left(\frac{d^2r}{dt^2}\right)^2 = 2\frac{dr}{dt}\frac{d^2r}{dt^2} = -\frac{2GM}{r^2}\frac{dr}{dt}.$$

integrate both sides with respect to time. Remember that we start at an initial radius of R and finish at an arbitrary radius r. This fixes the integration constant.

$$\left(\frac{d^2r}{dt^2}\right)^2 = -\int_0^t \frac{2GM}{r^2} \frac{dr}{dt} dt = -\int_R^r \frac{2GM}{r'^2} dr' = 2GM \left[\frac{1}{r} - \frac{1}{R}\right].$$

This is an infall process so we chose the negative branch of the square root (i.e. v < 0) which leaves

$$\frac{dr}{dt} = -\sqrt{2GM} \left(\frac{1}{r} - \frac{1}{R}\right)^{1/2}$$

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Separation of variables does the trick:

$$\tau_{\rm ff} = \int_0^{\tau_{\rm ff}} dt = -\int_R^0 \frac{dr}{\sqrt{2GM} \left(\frac{1}{r} - \frac{1}{R}\right)^{1/2}} = \frac{R^{3/2}}{\sqrt{2GM}} \int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x}} \, .$$

We have here made the substitution $x \equiv r/R$ and the final integrand evaluates to $\pi/2$, thus the exact result is

$$\Rightarrow \quad \tau_{\rm ff} = \frac{R^{3/2}}{\sqrt{2GM}} \frac{\pi}{2} = \sqrt{\frac{3\pi}{32G\rho_0}} \,. \tag{2}$$

A2: Shortcut with OoMA

Look separately at the left hand side (LHS) and right hand side (RHS) of Eq. 1. Make approximations and drop all constants of order unity. We denote an average velocity or acceleration by capital letters.

LHS:
$$v = \frac{dr}{dt} \sim V \sim \frac{R}{T} \Rightarrow A = \frac{d^2r}{dt^2} = \frac{dv}{dt} \sim -\frac{V}{T} \sim -\frac{R}{T^2} \sim -\frac{R}{\tau_{\rm ff}}$$

and RHS: $-\frac{4}{3}\pi G\rho r \sim -G\rho_0 R$.
is all together we get

Putting thi

$$-\frac{R}{\tau_{\rm ff}}\approx -G\rho_0 R$$

or

$$\tau_{\rm ff} \approx \frac{1}{\sqrt{G\rho_0}} \,. \tag{3}$$

Comparing this with Eq. 2 reveals that we are off (high) by a factor of $\sqrt{32/3\pi} = 1.84 < 2$.

So what's the typical free-fall time for the sun?

It is quicker to use the OoMA result:

$$\tau_{\rm ff,\,\odot} \approx \frac{1}{\sqrt{G\rho_{\odot}}} \approx \frac{1}{\sqrt{(2/3 \times 10^{-7} {\rm cm}^3 {\rm g}^{-1} {\rm s}^{-2})(1.4 {\rm g/cm}^3)}} \approx 3,000 {\rm s} \approx 1 {\rm hr}$$

The exact solution would have given us about 30 minutes. An interesting result!