AST 353 Astrophysics — Lecture Notes

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WHITE DWARF STARS (CONT.)

Review

The following properties about White Dwarfs (WDs) are useful to remember:

$$\boxed{M_{\rm WD} \sim 1 M_\odot}$$

and

$$\boxed{R_{\rm WD} \sim \frac{1}{100} R_\odot \sim 10,000 \text{ km}}\,,$$

which is much larger than the radius required to form a black hole (BH)

$$\boxed{R_{S,\odot} = \frac{2GM_\odot}{c^2} \approx 3 \text{ km} \quad \text{(Schwarzschild radius)} \quad .}$$

Non-Relativistic White Dwarfs

The pressure support required for a WD in HSE is

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V} \approx \frac{1}{3}\frac{GM^2}{R^4}\,. \tag{1}$$

The pressure available from NR degenerate (QM) electrons is

$$\langle P \rangle = K_{\rm NR} \langle n_e \rangle^{5/3}\,. \tag{2}$$

However, the electron number density n_e can be related to the mass density if we assume two nucleons per electron, as is the case for helium, carbon, and oxygen:

$$\rho \approx 2 m_H n_e\,. \tag{3}$$

Thus, in terms of mass density we have

$$\frac{K_{\rm NR}}{(2m_H)^{5/3}} \langle \rho \rangle^{5/3} \approx \frac{1}{3}\frac{GM^2}{R^4}$$

but the density also scales as $\langle \rho \rangle \propto M/R^3$ so the overall relation gives

$$\langle \rho \rangle^5 \propto \left(\frac{M^2}{R^4}\right)^3 = \frac{M^6}{R^{12}} \quad \text{and} \quad \langle \rho \rangle^5 \propto \left(\frac{M}{R^3}\right)^5 = \frac{M^5}{R^{15}}\,.$$

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A quick glance (divide by M^5 and multiply by R^{12}) gives that $M \propto 1/R^3$ or as a mass relation:

$$R \propto M^{-1/3}\,.$$

The constant of proportionality may be found by considering a solar size WD:

$$\boxed{R = \frac{R_\odot}{100}\left(\frac{M}{M_\odot}\right)^{-1/3}.} \tag{4}$$

Putting this into the original expression yields the relation:

$$\langle \rho \rangle \propto M^2$$

or an order of magnitude equation of

$$\boxed{\langle \rho \rangle \approx 10^6 \left(\frac{M}{M_\odot}\right)^2 \ \text{g cm}^{-3}\,.} \tag{5}$$

As the mass and density go up the electrons are forced into higher and higher momentum states and soon become relativistic. The critical density where the transformation between NR and UR regimes happens is about 10^6 g/cm^3. Past this point the star becomes unstable! We therefore expect the mass limit to be on the order of a solar mass $M_\odot$, which we shall now demonstrate.

Ultra-Relativistic White Dwarfs

We need to use a relativistic density to find the critical mass M_{crit}:

$$\text{As} \quad M_{\text{WD}} \to M_{\text{crit}} \qquad \Longrightarrow \qquad \langle P \rangle \to \langle P_{\text{UR}} \rangle = K_{\text{UR}} n_e^{4/3} = \frac{K_{\text{UR}}}{(2m_H)^{4/3}} \langle \rho \rangle^{4/3}\,,$$

where

$$K_{\text{UR}} \equiv \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}.$$

Recall that to counter-balance gravity we need

$$\langle P_{\text{UR}} \rangle \gtrsim \frac{1}{3}\frac{GM^2}{R^4}\,,$$

where here we also use $\langle \rho \rangle \sim M/R^3$ so that $\langle \rho \rangle^{4/3} \sim M^{4/3}/R^4$. Therefore, the radius cancels!

$$\frac{K_{\text{UR}}}{(2m_H)^{4/3}}\frac{M^{4/3}}{R^4} \gtrsim \frac{1}{3}\frac{GM^2}{R^4}\,,$$

so that

$$M \lesssim \left(\frac{3}{8}\right)^{3/2}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(2m_H)^2}\,.$$

This result is so important we write it again!

$$\boxed{M_{\text{ch}} = \frac{1}{4}\left(\frac{3}{8}\right)^{3/2}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{m_H^2} \approx 1.7\ M_\odot \qquad \text{(Chandrasekhar limit)}\,.} \tag{6}$$

A more precise calculation would actually yield the famous result:

$$\boxed{M_{\rm ch} \approx 1.4\ M_\odot \qquad \text{(precise Chandrasekhar limit)}\,.} \tag{7}$$

The meaning of this limit is that 'no WD can exist with $M > M_{\rm ch}$!

We will now rephrase this limit in terms of the "Planck mass"

$$\boxed{m_{\rm Pl} \equiv \left(\frac{hc}{G}\right)^{1/2} \approx 5\times 10^{-5}\ \text{g} \approx 10^{19}\ \text{GeV/c}^2 \qquad \text{(Planck mass)}\,.} \tag{8}$$

This is a modest mass! We usually think of 'Planck' quantities as being extreme scales where classical physics breaks down and we need a quantum theory of gravity. So if this is not a 'smallest mass' what is it? The conversion to energy gives a hint. Compare this with the highest energy obtained by the Large Hadron Collider (LHC) ~ 1000 GeV!

Meaning: The Planck mass is the smallest mass BH that can still be described by (classical) GR. To be explicit we cannot have the quantum wavelength be greater than the Schwarzschild radius!

$$\Longrightarrow \quad R_S \gtrsim \lambda_{\rm quantum}\,.$$

Here the quantum wavelength is not the "de Broglie wavelength" because if we think of the black hole as being a stationary particle then because the momentum is zero, $\lambda_{\rm dB} = h/p \to \infty$, which is not very helpful. Rather we use the "Compton wavelength" which is derived by considering the rest energy: $m_0c^2 \sim \epsilon \sim hc/\lambda_c$. We emphasize this wavelength:

$$\boxed{\lambda_c = \frac{h}{m_0 c} \qquad \text{(Compton wavelength)}\,.} \tag{9}$$

Thus, for a black hole of mass $M_{\rm BH}$ we get:

$$R_S \approx \frac{GM_{\rm BH}}{c^2} \gtrsim \frac{h}{M_{\rm BH}c} = \lambda_{\rm quantum} \quad \Longrightarrow \quad \boxed{M_{\rm BH} \gtrsim \sqrt{\frac{hc}{G}} = m_{\rm Pl}\,.} \tag{10}$$

Finally, we can recognize that the Chandrasekhar mass limit has this fundamental quantity built inside. WDs know about fundamental quantum gravity! We present a final casting of $M_{\rm ch}$:

$$\boxed{M_{\rm ch} \approx \frac{1}{4}\left(\frac{3}{8}\right)^{3/2} \frac{m_{\rm Pl}^3}{m_H^2} \sim \frac{m_{\rm Pl}^3}{m_H^2}\,.} \tag{11}$$