AST 353 Astrophysics — Lecture Notes

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WHITE DWARF STARS

Group Project – Calculus of Variations

Problem set 1 solutions are online as well as the introduction and description for the first group project. The goal is to get a handle on the sophisticated (an quite beautiful) mathematics of Calculus of Variations, which may be summarized by the Euler-Lagrange (E-L) equations. There are two parts:

(i) Group examinations of the readings.

(ii) A puzzle section to work out details for the Brachistochrome (shortest time) problem.

Intro to White Dwarfs

White Dwarfs (WDs) were serendipitously discovered by Alvan Clark in the mid 19^{th} Century. (Neutron stars on the other hand were theorized before they were discovered.) Careful observations revealed that Sirius was actually a double star system! The smaller and dimmer Sirius B was found to have very weird properties. The mass and radius gave a surprisingly high density:

$$M \sim 1 M_\odot \qquad \text{and} \qquad R \sim \frac{1}{100} R_\odot \qquad \Rightarrow \qquad \langle \rho \rangle \sim 10^6 \text{g/cm}^3 \,.$$

This is about a million times the average density of the sun! Thus, because these parameters are typical of many (dying) stars we have a new class of objects of hot (white), small (dwarf) stars.

Estimating the strength of gravity

We want to ask if Newtonian gravity is sufficient to describe WDs or if we need to turn to the corrections of Einstein's theory of general relativity. To do this we consider the important concept in general relativity (GR) of the Schwarzschild radius R_S. We derive this 'radius' with the extreme Newtonian escape velocity, which turns out to be right for the wrong reason! The escape velocity v_{esc} is the minimal velocity required to escape the influence of gravity. We know that the total energy is conserved and is zero because both the kinetic ($v \to 0$) and potential ($1/r \to 0$) energy go to zero at infinity:

$$E = \frac{1}{2} m v_{\text{esc}}^2 - \frac{GMm}{r} = 0 \,.$$

A black hole is an object that not even light can escape from so we can replace v_{esc} with c and find the radius at which gravity overcomes all else:

$$\boxed{R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)}\,.} \tag{1}$$

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This turns out to be the exact result from general relativity as well! But in GR the meaning of R_S does not have the familiar meaning our intuition suggests. Rather it is the radius in the sense of the circumference $C_{\rm EH}$ of the event horizon which is meaningful because the radial direction is 'warped' by the singularity at the 'center' of the black hole. In fact, the geometry gives a radius of

$$R_S = \frac{C_{\rm EH}}{2\pi}\,.$$

Hence, to gauge the importance of GR we can simply take the ratio of the object's Schwarzschild radius to its actual radius, where $\boxed{R_{S,\odot} \approx 3\text{ km.}}$

- **Sun** : $\dfrac{R_{S,\odot}}{R_\odot} \sim 10^{-6}$ — Small but measurable effect.
- **WD** : $\dfrac{R_{S,WD}}{R_{WD}} \sim \dfrac{1\text{ km}}{10,000\text{ km}} \sim 10^{-4}$ — GR is still unimportant.
- **NS** : $\dfrac{R_{S,NS}}{R_{NS}} \sim \dfrac{3\text{ km}}{10\text{ km}} \sim 0.5$ — GR is crucial for neutron stars.
- **BH** : $\dfrac{R_{S,BH}}{R_{BH}} \equiv 1$ — Redundant but instructive.

Basic Properties of White Dwarfs

For a WD to be in HSE we need

$$\langle P\rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V} \approx \frac{1}{3}\frac{GM^2}{R^4}\,.$$

Now for a (not too large) WD pressure is due to NR degenerate (QM) electrons. The pressure is from the electrons because their small masses mean they have large overlapping wave functions and therefore Fermi pressure. We can show that if WDs are more massive then we run into stability problems, but under about 10^6 g/cm^3 we are still in the NR regime.

The pressure is

$$\langle P\rangle = K_{\rm NR}\langle n_e\rangle^{5/3}\,, \tag{2}$$

where n_e is the electron number density.

Q: How do we relate n_e to the mass density ρ?
A: The WD mass is mainly provided by helium He, carbon C, and oxygen O. Thus, for completely ionized nuclei we have two nucleons (protons and neutrons) per free electron so to good approximation we have

$$\boxed{\rho \approx 2m_H n_e\,,} \tag{3}$$

where m_H is the mass of the hydrogen atom.