AST 353 Astrophysics — Lecture Notes

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CLASSICAL/QUANTUM GASES (CONT.)

Review

QM effects kick in when

$$\boxed{\ell \lesssim \lambda_{\text{dB}} \equiv \frac{h}{p} \qquad (\text{“de Broglie wavelength”})\,.} \tag{1}$$

The Fermi momentum is

$$\boxed{p_{\text{F}} = h\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}\,.} \tag{2}$$

The reader might not be impressed with this equation, but the Fermi momentum holds all the information we need! We don't need to memorize this formula but we should be able to derive it! The essential first step is

$$N = 2\frac{\text{volume of phase space}}{\text{volume of one cell } (\sim h^3)}\,.$$

Quantum Pressure

Now let's apply the concepts used to derive Eq. 2 to find the pressure under complete degeneracy. The kinetic energy for a NR gas goes as $\epsilon_{\text{kin}} = p^2/2m_0$. Also recall that the pressure is

$$P = \frac{2}{3}u_{\text{kin}} = \frac{2}{3}n\langle\epsilon_{\text{kin}}\rangle = \frac{n}{3}\frac{\langle p^2\rangle}{m_0}\,.$$

We expect the average momentum $\langle p\rangle$ to be roughly the Fermi momentum p_{F}. The exact relation is aided by recalling how moments (monomial averages) are made in continuous distribution statistics:

$$\langle p^n\rangle \equiv \frac{\int p^n\,dV}{\int dV} = \frac{\int_0^{p_{\text{F}}} 4\pi p^{n+2}\,dp}{\int_0^{p_{\text{F}}} 4\pi p^2\,dp} = \left(\frac{4\pi}{n+3}p_{\text{F}}^{n+3}\right) \Big/ \left(\frac{4\pi}{3}p_{\text{F}}^3\right) = \frac{3}{n+3}p_{\text{F}}^n\,. \tag{3}$$

The momentum volume integral involved a spherical integral out to the Fermi momentum p_{F}. By Eq. 3 the average squared momentum is $\langle p^2\rangle = \frac{3}{5}p_{\text{F}}^2$. Recall the relationship between momentum and number density from Eq. 2 of $p_{\text{F}} \sim hn^{1/3}$, which gives a pressure of

$$\boxed{P = \frac{h^2}{5m_0}\left(\frac{3}{8\pi}\right)^{2/3} n^{5/3} = K_{\text{NR}}n^{5/3} \qquad (\text{NR} + \text{QM})\,.} \tag{4}$$

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How does this change in the UR regime? We use the exact same reasoning but this time the kinetic energy for an UR gas is roughly $\epsilon_{\rm kin} \approx pc$. Thus, the pressure is

$$P = \frac{1}{3} u_{\rm kin} = \frac{1}{3} n \langle \epsilon_{\rm kin} \rangle = \frac{n}{3} \langle pc \rangle \, .$$

Finally, using Eq. 3 to rewrite the average squared momentum as $\langle p \rangle = \frac{3}{4} p_{\rm F}$ and Eq. 2 to replace the Fermi momentum $p_{\rm F} \sim h n^{1/3}$ gives the following relationship for UR pressure:

$$\boxed{P = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3} n^{4/3} = K_{\rm UR} n^{4/3} \qquad ({\rm UR} + {\rm QM}) \, .} \tag{5}$$

Note: The density dependence goes from $n^{5/3} \to n^{4/3}$ when moving to the relativistic regime. Values in between are described by an adiabatic index Γ so we can relate the pressure P to density ρ via the polytropic equation $P = K \rho^{\Gamma}$. Also, the physical constants are a clear indicator of what kind of physics to expect. If planck's constant h appears then we are dealing with quantum mechanics and if the speed of light c appears then we have introduced relativity! Furthermore, the relativistic case ceases to have any dependence on the (electron) mass.

Quantum NR/UR Transition (or Boundary)

So what is the boundary between the NR and UR gases? Remember that the Fermi momentum $p_{\rm F}$ is the maximum momentum of a (completely) degenerate gas. The semi-classical limit is found by assuming the velocity in the classical momentum goes to the speed of light and that momentum is the Fermi momentum:

$$p = mv \quad \to \quad p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \approx m_0 c \qquad \text{and} \qquad p \quad \to \quad p_{\rm F} = h \left(\frac{3}{8\pi} \right)^{1/3} n^{1/3} \, .$$

Equating these quantities gives the number density $n_{\rm UR}$ when the gas is first in the UR regime:

$$\boxed{n_{\rm UR} = \frac{8\pi}{3} \left(\frac{mc}{h} \right)^3 \approx 10 \left(\frac{mc}{h} \right)^3 \, .} \tag{6}$$

If we consider the gas to be made up of many independent species of particles (normally electrons, protons, and neutrons) the above equation says the lowest mass particle will become degenerate the soonest (i.e. at the lowest density). In fact, because the protons are still NR we can safely ignore any pressure from protons (i.e. $P \approx P_e$). Let's consider electrons!

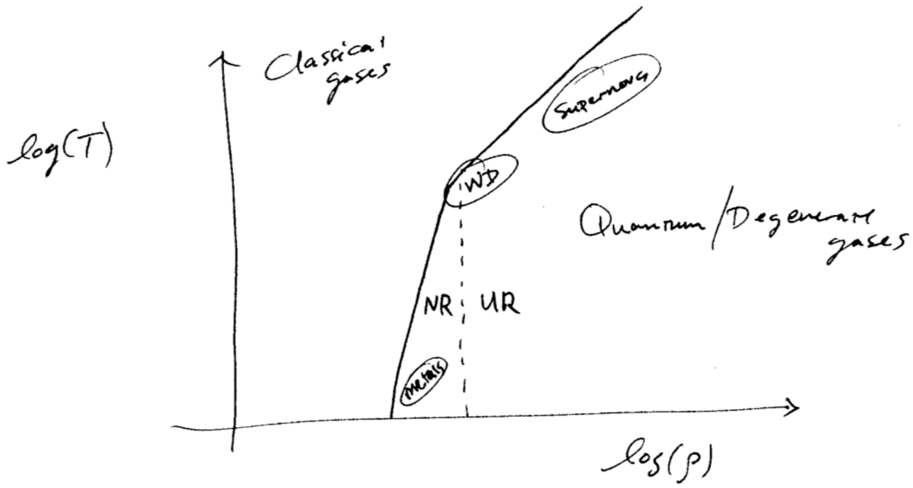
The mass of an electron is $m_e = 9.1 \times 10^{-28}$ g, which gives the following critical electron number density n_e to cross into the UR regime:

$$n_{\rm UR} \sim 10 \left(\frac{m_e c}{h} \right)^3 \sim 10 \left(\frac{10^{-27} \ {\rm g} \cdot 3 \times 10^{10} \ {\rm cm/s}}{(2/3) \times 10^{-26} \ {\rm g/s}} \right)^3 \sim 10^{30} \ {\rm cm}^{-3} \, .$$

To find a mass density we assume there are as many protons as electrons so that

$$\boxed{\rho_{\rm UR} \sim m_H \cdot n_{\rm UR} \sim 10^6 \ {\rm g} \ \cdot \ {\rm cm}^{-3} \, .} \tag{7}$$

This is about a million times the typical solar density (~ 1.4 g/cm^3) or the average density of the Earth (~ 5 g/cm^3). Astronomers were puzzled by this until QM was discovered in the 1920s.



White Dwarfs lie on the degenerate border between NR and UR gases, making them in danger of being unstable. Indeed, if a WD's mass goes slightly above $\sim 1.4 M_{\odot}$ a SN explosion occurs.