AST 353 Astrophysics — Lecture Notes

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HAWKING RADIATION

Hawking noticed that the area of a black hole always increases and linked this to the second law of thermodynamics. The rough idea is summarized as

$$\boxed{\frac{dA_{\rm BH}}{dt} \geq 0 \qquad \text{and} \qquad \frac{dS_{\rm BH}}{dt} \geq 0 \qquad \Rightarrow \qquad S_{\rm BH} = K A_{\rm BH}\,.} \tag{1}$$

The constant K can be roughly deduced from dimensional analysis:

$$S = k_{\rm B} \log W \propto k_{\rm B} \qquad \text{and} \qquad A_{\rm BH} \propto \ell_{\rm Pl}^2 \qquad \Rightarrow \qquad K = \frac{S_{\rm BH}}{A_{\rm BH}} \propto \frac{k_{\rm B}}{\ell_{\rm Pl}^2}\,.$$

Recall: The Planck length $\ell_{\rm BH}$ is the only combination of G, h, and c to give dimensions of length:

$$[G] = \frac{\text{cm}^3}{\text{g s}^2}\,, \qquad [h] = \frac{\text{g cm}^2}{\text{s}}\,, \qquad \text{and} \qquad [c] = \frac{\text{cm}}{\text{s}} \qquad \Rightarrow \qquad \left[\frac{Gh}{c^3}\right] = \text{cm}^2\,.$$

Jacob Bekenstein first worked out the black hole thermodynamics. He had to use the full quantum field theory to get the exact constant of proportionality which adds a factor of $\frac{1}{4}$:

$$\boxed{S_{\rm BH} = \frac{k_{\rm B}}{4}\frac{A_{\rm BH}}{A_{\rm Pl}}\,,} \tag{2}$$

where the Planck length is $\ell_{\rm Pl} = ct_{\rm Pl} \sim 10^{-33}$ cm and the Planck area is $A_{\rm Pl} = \ell_{\rm Pl}^2 \sim G\hbar/c^3 \sim 10^{-66}$ cm^2. The area of a black hole is $A_{\rm BH} = 4\pi R_{\rm S}^2$, which is valid even in the curved spacetime around black holes because r is a circumference type coordinate. Therefore, the entropy of a black hole is given by

$$\boxed{S_{\rm BH} = k_{\rm B}\frac{4\pi G}{\hbar c}M^2 \approx k_{\rm B}10^{77}\left(\frac{M}{M_\odot}\right)^2\,.} \tag{3}$$

Compare this to the current entropy of the sun $S_\odot \approx 10^{58}\ k_{\rm B}$. Thus, black holes are a huge source of entropy, the largest in the Universe!

Caveat: If black holes have entropy they must also have non-zero temperature! Recall the definition of entropy from the first law of thermodynamics ($dE = TdS - PdV + \mu dN$):

$$S \equiv \frac{dQ}{T}\,.$$

The energy transfer must come from the mass of the black hole so we combine

$$T_{\rm BH}dS_{\rm BH} = dQ = d\left(Mc^2\right) = c^2 dM \qquad \text{and} \qquad dS_{\rm BH} = k_{\rm B}\frac{8\pi G}{\hbar c}MdM$$

to obtain the formal definition of the temperature of a black hole:

$$\boxed{T_{\rm BH} = \frac{\hbar c^3}{8\pi G k_{\rm B}}M^{-1} \approx 10^{-7}\ \text{K}\left(\frac{M}{M_\odot}\right)^{-1}\,.} \tag{4}$$

What does this mean? We have two seemingly contradictory statements: (i) Any body with non-zero temperature emits radiation but (ii) Black holes are not supposed to let anything escape! The answer is that the second sentence really only applies to classical GR. When we combine classical GR with quantum mechanics we get a completely different theory where BHs may even "radiation!"

The Hawking Effect

In quantum mechanics the rule is whatever is not strictly forbidden may happen. Thus, we consider particle-antiparticle pair-creation near the event horizon. For a small amount of time a virtual particle pair exists near the horizon and quickly disappear again. However, this happens often enough that one particle may enter the black hole while the other escapes! This is how a "real" particle may be formed and emitted. To place this on a quantitative footing we consider energy conservation. The real particle on the outside of the horizon has positive energy $\epsilon_1 = \epsilon > 0$ while the particle which falls into the black hole has a negative energy $\epsilon_2 = -\epsilon < 0$. We may estimate the "black hole luminosity" because the quantum mechanical effects predict we are close to equilibrium, or the radiation is that of a black-body,

$$\boxed{L_{\rm BH} = 4\pi R_{\rm S}^2 \sigma_{\rm SB} T_{\rm BH}^4 \approx 10^{-20}\ {\rm erg\ s^{-1}} \left(\frac{M}{M_\odot}\right)^{-2} \approx 10^{-54}\ L_\odot \left(\frac{M}{M_\odot}\right)^{-2}.} \tag{5}$$

This is incredibly low(!), which makes sense as we expect a classical 'black hole' if $M = M_\odot$.
Note: A smaller mass M produces a higher luminosity $L_{\rm BH}$. The smallest mass of a classical black hole is the Planck mass $m_{\rm Pl}$ so the highest luminosity is roughly $10^{22}\ L_\odot$, but this would only last for a timescale on the order of the Planck time.

Black hole evaporation

The Hawking time $\tau_{\rm Hawk}$ is the time required to completely evaporate a black hole. To estimate $\tau_{\rm Hawk}$ consider that luminosity is simply the time derivative of the energy:

$$\frac{d\left(Mc^2\right)}{dt} = -L_{\rm BH} \propto M^{-2}\,.$$

This may be solved by separation of variables to obtain

$$\boxed{\tau_{\rm Hawk} \approx 10^{66}\ {\rm yr} \left(\frac{M(t=0)}{M_\odot}\right)^3 \approx 10^{-54}\ L_\odot \left(\frac{M}{M_\odot}\right)^{-2}.} \tag{6}$$

Note: Compared to the Hubble time $t_{\rm H} \approx 14$ Gyr the Hawking time for a solar mass black hole $\tau_{{\rm Hawk},\odot}$ is forever! Therefore, the effect is negligible for astrophysical black holes. However, there is strong dependence on mass so what about low mass black holes? If we set $\tau_{\rm Hawk} \sim t_{\rm H}$ then we obtain the mass for a "mini-BH," which would be evaporating right now if it was formed right after the Big Bang:

$$\boxed{M_{\rm mini-BH} \approx 10^{14}\ {\rm g}\,.} \tag{7}$$

A few decades ago gamma ray bursts (GRBs) were discovered and postulated to be these "mini-BHs," however, we now know that GRBs are normal stars in the final stages of collapse before forming a black hole. The jury is out on "primordial-BHs" or "laboratory-BHs" but it is doubtful that we need to worry about these cases. The energy of a Planck mass black hole is roughly $\epsilon_{\rm Pl} \sim m_{\rm Pl}c^2 \sim 10^{19}$ GeV, which is far outside the range of the LHC.

Information content of black holes

Black holes are very good at hiding information, which is why they have such high entropy.
Q: How many possible progenitor configurations could a black hole have had? or how many states W would need to be considered for a solar mass black hole?
A: For $M_{\mathrm{BH}} \sim M_{\odot}$ we have $S_{\mathrm{BH},\odot} \approx k_{\mathrm{B}}\ 10^{77} = k_{\mathrm{B}} \log W$. Thus, the enormous number of states is

$$W \approx \exp\left(10^{77}\right) \sim 10^{10^{77}}\,.$$

Exam Review

Please know the following formulae:

- Schwarzschild Metric and Schwarzschild Radius R_{S}
- "Constants of motion" machinery including the Euler-Lagrange equation
- Q7: $\tau_{\mathrm{BH}} \sim R_{\mathrm{S}}/c$
- Q8: Dimensional analysis (e.g. $S = KA$)
- Taylor expansions, esp. $(1+x)^n \approx 1 + nx$ if $x \ll 1$
- Anything not here will be given on the exam (ask if you are unsure)

Thank you for a great semester!