AST 353 Astrophysics — Lecture Notes

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BLACK HOLES

Review Quiz 7

The fate of the universe is determined by its mass. We characterize this by a non-dimensional parameter called the fractional density $\Omega \equiv \rho/\rho_{\text{crit}}$. If $\Omega > 1$ then the universe will collapse back on itself – a process called the "Big Crunch." Thus, an answer to the quiz question of how long it would take for the 'crunch' to happen is

$$\tau_{\text{BH}} \equiv \frac{R_{\text{S}}}{c} \approx \frac{R_{\text{H}}}{c} = H_0^{-1} \sim 14 \text{ Gyr},$$

where R_{H} is the Hubble radius, or roughly the size of the observable universe.

The story told by the observer at the surface

Oppenheimer and Snyder modeled the collapse of massive stars. Their solution is

$$\boxed{\tau(r) = \frac{1}{2}\frac{R_{\text{S}}}{c}\left(\frac{r_0}{R_{\text{S}}}\right)^{3/2}(\sin\eta + \eta) \qquad \text{where} \qquad \eta = \cos^{-1}\left(2\frac{r}{r_0} - 1\right).} \tag{1}$$

This is much easier to visualize with a plot. We may use the example of $r_0 = 5\ R_{\text{S}} \Rightarrow \tau_0 = 17.6 R_{\text{S}}/c$. **Note:** Observer ① riding on the surface of the collapsing star does **not** notice anything unusual when crossing the event horizon! They reach the central singularity in finite time.

The story told by the observer at infinity

As before we consider a "radial plunge" toward the black hole but now we need to know the radius as measured by the distant observer. Here $r = r(t)$ where t is the coordinate time measured by observer ②. The relation is found by combining the $\frac{dr}{d\tau}$ result with conservation of energy:

$$\text{Combine:} \qquad \begin{cases} \dfrac{dr}{d\tau} = -c\sqrt{\dfrac{R_{\text{S}}}{r} - \dfrac{R_{\text{S}}}{r_0}} \\[2ex] e = c^2\left(1 - \dfrac{R_{\text{S}}}{r}\right)\dfrac{dt}{d\tau} = c^2\sqrt{1 - \dfrac{R_{\text{S}}}{r_0}} \end{cases}$$

The second equation for e came from the initial condition $\dot{r}(r_0) = 0$. After a little algebra we get

$$\boxed{\frac{dr}{dt} = \frac{dr}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1} = -c\frac{\sqrt{r_0/r - 1}}{\sqrt{r_0/R_{\text{S}} - 1}}\left(1 - \frac{R_{\text{S}}}{r}\right).} \tag{2}$$

However, this is fairly complicated and can be solved with much less work if we are only interested in the behavior near the Schwarzschild radius. The approximation near the horizon is

$$\boxed{\frac{dr}{dt} \approx -c\left(1 - \frac{R_{\text{S}}}{r}\right) \approx -\frac{c}{R_{\text{S}}}(r - R_{\text{S}}).} \tag{3}$$

The solution is obtained from separation of variables:

$$\frac{dr}{r - R_{\rm S}} \approx -\frac{c}{R_{\rm S}} dt \qquad \Rightarrow \qquad \boxed{r(t) \approx R_{\rm S} + b \exp\left[-\frac{ct}{R_{\rm S}}\right]}$$

where b is some constant whose exact value is unimportant.

Note: From the viewpoint of the distant observer the radius approaches the event horizon only assymptotically and would take an infinite amount of time to actually cross it. Thus, the collapse appears to "freeze" at the horizon.

Relativists found this solution first and thought "Surely, this is not physical so black holes cannot form!" However, they soon realized this is an optical illusion. But if reality is split then which frame represents the 'true' physics? They both do, but one is the experience of the star which forms a black hole singularity and the other is an observation of the physics from a source which is an information sink. The physics for the distant observer is inherently limited because no information inside the event horizon can escape.

Black holes have 'no hair'

Eventually people figured out that black holes are quite simple! They are like elementary particles in that they are identical and can be described by only a few paramters:

(i) Mass – M (ii) Angular momentum – J (iii) Charge – Q.

Astrophysical black holes cannot hold a charge because they will quickly attract particles of opposite charge and neutralize. Also if a black hole has any variation from these simple parameters (e.g. a bump in the event horizon) the differences will be radiated away through gravity waves.

Hawking Radiation

We would like to know how black holes evolve over time. Hawking's big idea was to consider the area of black holes before and after some physical process. For black hole mergers or mass accretion it was found that the area of the event horizon **always** increases!

$$\boxed{\frac{dA_{\rm BH}}{dt} \geq 0 \qquad \text{(Area-increase theorem).}} \tag{4}$$

The second law of thermodynamics dictates that the entropy of a closed system **always** increases!

$$\boxed{\frac{dS}{dt} \geq 0 \qquad (2^{\rm nd} \text{ law of thermodynamics}).} \tag{5}$$

Notice the curious similarity between the two laws!

This observation leads us to deduce the entropy of a black hole as

$$S_{\rm BH} \propto A_{\rm BH}\,,$$

where the constant of proportionality can be deduced from a dimensional analysis argument.