# AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith
(Dated: April 23, 2013)

## BLACK HOLES

### Review

We discovered the motion of particles in Schwarzschild geometry admits two constants of motion:

$$\boxed{e = \frac{\epsilon}{m_0} = c^2 \left(1 - \frac{R_S}{r}\right) \frac{dt}{d\tau} = \text{constant} \qquad (\text{“Energy per unit rest mass”}).} \tag{1}$$

$$\boxed{j = \frac{J}{m_0} = r^2 \frac{d\varphi}{d\tau} = \text{constant} \qquad (\text{“Angular momentum per unit rest mass”}).} \tag{2}$$

In order to deduce these we utilized spherical symmetry to choose the easiest path possible. Thus, we restricted to the equatorial plane ($\vartheta = \frac{\pi}{2}$ and $\dot{\vartheta} = 0$). Then we set our Lagrangian to be

$$L \equiv c\frac{d\tau}{d\tau} = L(t, \dot{t}, r, \dot{r}, \varphi, \dot{\varphi}) = \sqrt{c^2 \left(1 - \frac{R_S}{r}\right) \dot{t}^2 - \left(1 - \frac{R_S}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\varphi}^2}\,,$$

and the constants of motion (conserved quantities) follow from the Euler-Lagrange equations.

### Stellar Collapse

What happens when a star has a mass which is greater than the Oppenheimer-Volkoff limit ($M > M_{OV}$)? We do not know of a mechanism to ‘stop’ gravity at this point so we expect collapse! Indeed Oppenheimer and Snyder showed this in 1939. We follow their argument.

We must consider the collapse from the viewpoint of two different observers:

Observer ① is at the surface of the star and

Observer ② is very far from the star (i.e. at infinity).

The choice is a good one because observer ① knows everything from the star’s perspective (which of course is what we want to know) and they can still communicate with the observer ② at $r = \infty$. Furthermore, the geometry at observer ② is simple . . . it is flat! The result is that observer ② recieves signals at longer and longer wavelengths, or equivalently the temporal separation between photons is greater and greater.

The Schwarzschild metric is applicable to the **exterior** of collapsing stars even though the spacetime is **not** time-independent! This is a result of “Birkhoff’s Theorem” which allows us to treat the exterior as if the gravitational field was caused by a point mass. This is like drawing a Gaussian surface around a blob of mass (or charge) and ignoring all of the complications inside.

**The story told by the observer at the surface**

We are assuming spherical symmetry so we simply consider radial plunge orbits with $\dot{\vartheta} = \dot{\varphi} = 0$.

$$\boxed{-\frac{ds^2}{d\tau^2} = c^2 = c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\dot{t}^2 - \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}\dot{r}^2 \qquad (\text{“Radial plunge orbits”}).} \tag{3}$$

This is one equation with two unknowns so we need more information! Conservation of energy says

$$e = c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\frac{dt}{d\tau} = \text{constant} \qquad \Rightarrow \qquad \dot{t}^2 = \frac{e^2}{c^2}\left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}.$$

Thus, with a little algebra we obtain

$$\boxed{\frac{dr}{d\tau} = -c\left[\frac{e^2}{c^4} - 1 + \frac{R_\mathrm{S}}{r}\right]^{1/2}.} \tag{4}$$

We have $\dot{r} < 0$ because this is an inward plunge.
**Remember:** The coordinate $r(\tau)$ determines the surface of the collapsing star!
Now fix the constant $e$ so the plunge begins from rest, i.e. $\dot{r} = 0$ at $r_0$:

$$\begin{aligned} \Rightarrow \qquad 0 &= \frac{e^2}{c^4} - 1 + \frac{R_\mathrm{S}}{r} \\ \Rightarrow \qquad \frac{dr}{d\tau} &= -c\sqrt{\frac{R_\mathrm{S}}{r} - \frac{R_\mathrm{S}}{r_0}}\,. \end{aligned}$$

Solve by separation of variables:

$$\sqrt{\frac{r_0}{R_\mathrm{S}}}\int_{r_0}^{r}\frac{dr'}{\sqrt{r_0/r' - 1}} = -c\int_0^{\tau} d\tau'\,.$$

The integral is not immediately obvious but we can use the “cycloid parameter”

$$\eta = \cos^{-1}\left(2\frac{r}{r_0} - 1\right) \qquad \Rightarrow \qquad \frac{d\eta}{dr} = \frac{-1}{\sqrt{1 - (2r/r_0 - 1)^2}}\frac{2}{r_0} = -\frac{1}{r}\frac{1}{\sqrt{r_0/r - 1}}\,.$$

When we do this we immediately find that

$$\boxed{\tau(r) = \frac{1}{2}\frac{R_\mathrm{S}}{c}\left(\frac{r_0}{R_\mathrm{S}}\right)^{3/2}(\sin\eta + \eta) \qquad \text{where again} \qquad \eta = \cos^{-1}\left(2\frac{r}{r_0} - 1\right).} \tag{5}$$

Therefore, the time to reach the center (i.e. $\eta \to \pi$ as $r \to 0$) is

$$\tau_0 = \tau(r = 0) = \frac{\pi}{2}\frac{R_\mathrm{S}}{c}\left(\frac{r_0}{R_\mathrm{S}}\right)^{3/2}.$$

We put this into an order of magnitude form to estimate the collapse time of a black hole:

$$\boxed{\tau_\mathrm{BH} \equiv \frac{R_\mathrm{S}}{c} \approx 10^{-5}\ \mathrm{s}\ \left(\frac{M}{M_\odot}\right).} \tag{6}$$

**Note:** The collapse time for a black hole the size of the universe is the Hubble time $H_0^{-1} \approx 14$ Gyr.