AST 353 Astrophysics — Lecture Notes

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BLACK HOLES

Review

Recall last time we derived the Schwarzschild line element:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (\text{“Schwarzschild metric”}),} \tag{1}$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the angular part. The realm of validity requires that GR give the same answer as Newtonian gravity for weak fields. That is how we brought in the mass.

The other point we considered was that time slows down deep inside gravitational potentials. This phenomenon is perhaps best understood by the gravitational redshift equation:

$$\boxed{\nu_\infty = \nu_{\mathrm{em}} \sqrt{1 - \frac{2GM}{c^2 r}} \qquad (\text{“Gravitational redshift”}).} \tag{2}$$

If we do the perverse thing of emitting the photon at the Schwarzschild radius,

$$\boxed{r = R_{\mathrm{S}} = \frac{2GM}{c^2} \qquad (\text{“Schwarzschild radius”}),} \tag{3}$$

the observer at infinity sees a zero frequency photon, but this is really no photon at all and the source is truly “black” to the observer! The metric becomes singular at R_{S}, but this is a “coordinate singularity” because it can be transformed away by changing to a different set of coordinates. This is similar to the funny things that happen in spherical coordinates at $r = 0$ or $\vartheta = \pi$. However, the singularity at $r = 0$ for a black hole is real, and is the product of the limitations of classical relativity theory. We give extra credit to those who can discover the so called “theory of everything.”

Motion in Schwarzschild geometry

In general, particles move according to the forces which are present. If they are only influenced by gravity then they travel on spacetime geodesics. The path a particle takes is its world line, which maximizes the particle's own proper time. For $c^2 d\tau^2 = -ds^2$ we have geodesic motion when

$$-\int_A^B ds = c\int_A^B d\tau = c\int_A^B \left(\frac{d\tau}{d\tau}\right) d\tau = \text{extremal}.$$

This is the “principle of extremal time.” The last equality may seem trivial at this point but we do so to take advantage of the Euler-Lagrange formalism:

$$\int_A^B L d\tau = \text{extremal} \qquad \text{where} \qquad L \equiv c\frac{d\tau}{d\tau}\,.$$

For simplicity we only consider motion in the equatorial plane so $\vartheta = \frac{\pi}{2}$ and $d\vartheta = 0$. However, because of spherical symmetry this is not a real restriction and only makes the algebra nicer. We rewrite the line element after making this choice:

$$\boxed{c^2 d\tau^2 = -ds^2 = c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)dt^2 - \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1} dr^2 - r^2 d\varphi^2\,.} \tag{4}$$

Here we may divide by $d\tau^2$ to put everything in terms of proper velocities:

$$c^2 = c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\dot{t}^2 - \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2\,, \tag{5}$$

where $\dot{t} = dt/d\tau$, $\dot{r} = dr/d\tau$, and $\dot{\varphi} = d\varphi/d\tau$. The Lagrangian is then

$$\boxed{L \equiv c\frac{d\tau}{d\tau} = L(t, \dot{t}, r, \dot{r}, \varphi, \dot{\varphi}) = \sqrt{c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\dot{t}^2 - \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2}\,.} \tag{6}$$

Notice: The Lagrangian is always a constant but this is not an inconsistency, rather it represents the physical constraint that particles travel on timelike curves. We deduce the geodesics by first applying the E-L formalism and use the constraint after taking derivatives. We must solve:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t}\,, \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}\,, \quad \text{and} \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}\,.$$

However, L does not depend on two of the variables, i.e. $\partial L/\partial t = \partial L/\partial \varphi = 0$, so t and φ are "ignorable." Physically this means there are two conserved quantities or constants of motion.

Constants of motion

1. The time derivative is zero!

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} = 0 \qquad \Rightarrow \qquad \frac{\partial L}{\partial \dot{t}} = \text{constant} = \frac{c^2(1 - R_\mathrm{S}/r)\dot{t}}{L}\,,$$

but L is constant so we are left with

$$c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\frac{dt}{d\tau} = \text{constant}\,. \tag{7}$$

Q: What is the interpretation? **Recall:** A particle's energy in SR is $\epsilon = \gamma m_0 c^2$ where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor. We can rephrase the SR line element by letting $dx = v dt$:

$$-c^2 d\tau^2 = ds^2 = -c^2 dt^2 + dx^2 = -c^2 dt^2\left(1 - \frac{v^2}{c^2}\right) \qquad \Rightarrow \qquad \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma\,.$$

The result is that energy is actually $\epsilon = m_0 c^2 dt/d\tau$! Therefore, our constant of motion is

$$\boxed{e = \frac{\epsilon}{m_0} = c^2\left(1 - \frac{R_\mathrm{S}}{r}\right)\frac{dt}{d\tau} = \text{constant} \qquad \text{("Energy per unit rest mass").}} \tag{8}$$

2. The angular derivative is zero!

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} = 0 \qquad \Rightarrow \qquad \frac{\partial L}{\partial \dot{\varphi}} = \text{constant} = -\frac{r^2\dot{\varphi}}{L}\,,$$

but L is again constant so we are left with

$$r^2\frac{d\varphi}{d\tau} = \text{constant}\,. \tag{9}$$

Q: What is the interpretation? **Recall:** The angular momentum in Newtonian theory is $J = m_0 r v = m_0 r^2 \dot{\varphi}$. Here we used the fact that $v = \omega r = \dot{\varphi} r$. Therefore, our constant of motion is

$$\boxed{j = \frac{J}{m_0} = r^2\frac{d\varphi}{d\tau} = \text{constant} \qquad (\text{“Angular momentum per unit rest mass”}).} \tag{10}$$