AST 353 Astrophysics — Lecture Notes

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BLACK HOLES

Schwarzschild geometry (cont.)

Einstein's equation represents a complicated set of coupled ordinary differential equations so it is almost a miracle that there are any analytical solutions! The solutions necessarily have a number of simplifying assumptions, all of which were made by Schwarzschild to model spacetime outside stars. He said this is how to make things simple: (i) Assume a static spacetime $\Rightarrow$ no change in time $\Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial x^0} = 0$ (ii) Assume point masses $\Rightarrow$ spherical symmetry $\Rightarrow$ spherical coordinates.

For this reason we briefly review spherical coordinates. The coordinates themselves are

$$\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r \sin\vartheta \sin\varphi \\ z &= r \cos\vartheta \,. \end{aligned} \tag{1}$$

Thus, differential coordinate distances are derived from the product rule:

$$dx = \sin\vartheta \cos\varphi dr + r \cos\vartheta \cos\varphi d\vartheta - r \sin\vartheta \sin\varphi d\varphi \,,$$

and so on for y and z. If we went through the trouble of finding dy and dz and squaring them we would find the Pythagorean Theorem changes form:

$$d\ell^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \right) = dr^2 + r^2 d\Omega^2 \,,$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the angular or tangential part of the line element.
Note: $dV = dxdydz = r^2 \sin\vartheta dr d\vartheta d\varphi = dr dA$.

Physically, because the force components are only in the radial direction curvature is only possible in the radial direction. We keep the exact expression for radial curvature and gravitational redshift (i.e. time-dilation) as unknowns in the metric. These simplifications reduce the independent metric components from ten to two! The line element is

$$ds^2 = -A(r) c^2 dt^2 + B(r) dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \right) \,, \tag{2}$$

and the corresponding spherically symmetric, static metric is

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\vartheta \end{pmatrix} . \tag{3}$$

In vacuum the stress-energy tensor vanishes:

$$T_{\mu\nu} = 0 \qquad \Rightarrow \qquad \boxed{R_{\mu\nu} = 0 \,.} \tag{4}$$

We have to go back to the definition of the Ricci tensor to calculate all of the components, but schematically we get something like $R \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$, which involves first and second derivatives of the metric! With all of the derivatives we can simplify the algebra by using exponentials. We define $\nu(r) \equiv \log A(r)$ and $\lambda(r) \equiv \log B(r)$ so the line element becomes

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \right) \,. \tag{5}$$

The result is that the diagonal components of $R_{\mu\nu}$ are nonzero. If we drop the explicit dependence on r and denote radial derivatives by primes (i.e. $\nu' = d\nu/dr,\ \lambda' = d\lambda/dr,$ and $\nu'' = d^2\nu/dr^2$) then

$$\begin{aligned} R_{00} &= \frac{1}{2}e^{\nu-\lambda}\left[-\nu'' + \frac{1}{2}\lambda'\nu' - \frac{1}{2}\nu'^2 - \frac{2\nu'}{r}\right] \\ R_{11} &= \frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{\lambda'}{r} \\ R_{22} &= e^{-\lambda}\left(1 + \frac{1}{2}r\nu' - \frac{1}{2} + \lambda'\right) - 1 = R_{33}/\sin^2\vartheta\,. \end{aligned} \tag{6}$$

To solve this system of ordinary differential equations we use $R_{00} = R_{11} = 0$ to produce

$$\nu' + \lambda' = 0 \qquad \Rightarrow \qquad \nu(r) + \lambda(r) = K = \text{const}\,.$$

This is valid for all radii. In particular, as $r \to \infty$ spacetime is flat and we have the standard SR line element $ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$ which gives

$$e^\nu = e^\lambda = 1 \qquad \Rightarrow \qquad \nu = \lambda = 0 \qquad \text{(at infinity)}.$$

However, this is valid for all radii and we have established the value of $K = 0$ and found

$$\boxed{\nu(r) = -\lambda(r)\,.} \tag{7}$$

We now use $R_{22} = 0$ to derive a relation for $\nu(r)$

$$e^{-\lambda}\left[1 + \frac{1}{2}r\left(\nu' - \lambda'\right)\right] = e^\nu\left[1 + r\nu'\right] = \frac{d}{dr}\left(re^\nu\right) = 1\,.$$

Upon integration we find $re^\nu = r - C_0$ or $e^\nu = 1 - C_0/r$, where C_0 is a constant of integration.

Q: How do we fix the constant C_0? **A:** We already know that $g_{00} = -e^\nu$ is related to the gravitational redshift, so the equation must be valid for weak fields too: $\Rightarrow \quad -g_{00} = 1 + 2\varphi/c^2 = 1 - 2GM/c^2r$. This is valid at large radii where Newtonian theory holds, but it must also be valid at other radii because the equation is general. Therefore, the constant is set to be $C_0 = 2GM/c^2$ and the metric element is fixed to be

$$\boxed{e^\nu = 1 - \frac{2GM}{c^2r} = -g_{00}\,.} \tag{8}$$

The Schwarzschild metric is the solution of a static, spherically symmetric, vacuum spacetime:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\Omega^2 \qquad \text{(“Schwarzschild metric”)}.} \tag{9}$$

We think we have some intuition about these coordinates because they are labeled as the familiar $(t, r, \vartheta, \varphi)$ but they are not the same! The angular coordinates are still the same as there is no curvature in those directions, however, the radial coordinate is NOT the physical distance from the center. Instead it is related to a “circumference radius.” Consider the (proper) physical size of a surface at a coordinate r. Here $t =$ constant and $r =$ constant so the line element and area are

$$ds^2|_{t,r=\text{const}} = r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \qquad \Rightarrow \qquad A = \int dA = r^2\int_0^{2\pi}d\varphi\int_0^\pi \sin\vartheta d\vartheta = 4\pi r^2\,.$$

Flow of time in Schwarzschild geometry

Consider two stationary observers ($dr = d\vartheta = d\varphi = 0$) with one (A) at $r \to \infty$ and the second (B) at arbitrary r. To figure out the proper time each observer would measure we use the definition

$$\boxed{c^2 d\tau^2 = -ds^2\,.} \tag{10}$$

This is the time for the "true" physics. For observer (A) at $r \to \infty$ we have

$$ds^2 = -c^2 d\tau_\infty^2 = -c^2 dt^2$$

so the meaning of our coordinate time t is that it is the proper time of an observer at infinity. Now the proper time of observer (B) is affected by the gravitational redshift. We find

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2\,,$$

which allows a direct relation between the proper times in both frames:

$$\boxed{d\tau = d\tau_\infty \sqrt{1 - \frac{2GM}{c^2 r}}\,.} \tag{11}$$

Therefore, time slows down deep inside a gravitational potential well.

What is the effect on light? Well, similar to the pseudo-Newtonian case the light is redshifted. Observer (B) emits a photon with frequency $\nu_{\mathrm{em}} = 1/d\tau$ and observer (A) measures a different frequency $\nu_\infty = 1/d\tau_\infty$. The redshift is given as follows:

$$\boxed{\nu_\infty = \nu_{\mathrm{em}} \sqrt{1 - \frac{2GM}{c^2 r}} \qquad \text{(“Gravitational redshift”).}} \tag{12}$$

This relation reduces to the Newtonian case from before in a weak field $\nu_\infty \approx \nu_{\mathrm{em}}(1 - GM/c^2 r)$!

Note: Something happens at the special Schwarzschild radius

$$\boxed{r = R_{\mathrm{S}} = \frac{2GM}{c^2} \qquad \text{(“Schwarzschild radius”).}} \tag{13}$$

For light emitted from $r = R_{\mathrm{S}}$ the frequency is $\nu_\infty = 0$, no matter what frequency was originally emitted! But photons with zero frequency (or zero energy and infinite wavelength) cease to exist. If we cannot see it then this is truly a "black hole." No one wanted such a thing when the Schwarzschild solution was originally proposed but it unavoidably comes out in the math.