AST 353 Astrophysics — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith

(Dated: April 11, 2013)

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Schwarzschild geometry

The gravitational field **outside** a spherically symmetric object is characterized by the geometry of the spacetime in GR. We assume the object has a mass M contained in a radius R. We further assume that there is no time dependence so this is a "static field." Because of the symmetry the convenient coordinate system is spherical coordinates and the metric can only depend on the radial coordinate. The most general spacetime interval may be written as

$$ds^2 = -A(r)c^2dt^2 + B(r)dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) . \tag{1}$$

In order to figure out the exact form of the functions $A(r)$ and $B(r)$ we solve the Einstein field equation for the vacuum (i.e. $T_{\mu\nu} = 0$). We can do this because we are outside the star/object. Therefore the Field equation simplifies to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} = 0\,, \tag{2}$$

which we may rewrite as

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R\,.$$

However, this too can be simplified by taking the trace of both sides! To do this we recall the definition of the Ricci scalar as the trace of the Ricci tensor:

$$R \equiv g^{\mu\nu}R_{\mu\nu}\,.$$

Note: The trace of the metric equals the number of dimensions because $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$:

$$g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4\,.$$

We rewrite the trace of our equation as

$$\begin{aligned} \Rightarrow \quad g^{\mu\nu}R_{\mu\nu} &= \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R \\ \Rightarrow \quad R &= 2R \\ \Rightarrow \quad R &= 0\,. \end{aligned}$$

Therefore, the general vacuum solution of the Einstein equation (Eq. 2) is

$$\boxed{R_{\mu\nu} = 0 \qquad \text{(Vacuum solution of the Einstein equation).}} \tag{3}$$