AST 353 Astrophysics — Lecture Notes

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NEUTRON STARS (CONT.)

Recall that the Schwarzschild radius $R_S = \frac{2GM}{c^2}$ is used to gauge the importance of GR:

- **Sun** : $\dfrac{R_{S,\odot}}{R_\odot} \sim 10^{-6}$ Small but measurable effect.
- **WD** : $\dfrac{R_{S,WD}}{R_{WD}} \sim \dfrac{1 \text{ km}}{10,000 \text{ km}} \sim 10^{-4}$ GR is still unimportant.
- **NS** : $\dfrac{R_{S,NS}}{R_{NS}} \sim \dfrac{6 \text{ km}}{10 \text{ km}} \sim 0.6$ GR is crucial for neutron stars.

Describe the mechanical structure with the Oppenheimer–Volkoff equation

The Oppenheimer–Volkoff equation was derived to be

$$\boxed{\frac{dP}{dr} = -\rho \frac{Gm}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \qquad \text{(Oppenheimer–Volkoff).}} \tag{1}$$

We now consider the idealized case of constant density

$$\rho = \rho_0 = \text{constant}, \tag{2}$$

which is the profile for an "incompressible" gas. Of course the pressure will still have a gradient so this gives good qualitative results. First we need the mass coordinate

$$m = m(r) = \frac{4\pi}{3}\rho_0 r^3 \qquad \Rightarrow \qquad M = m(R) = \frac{4\pi}{3\rho_0 R^3}\,. \tag{3}$$

To solve the equation we use separation of variables. The idea is based on the realization that the right hand side separates into functions of the independent and dependent variables:

$$\frac{dP}{dr} = \frac{g(r)}{f(P)} \qquad \Rightarrow \qquad \int f(P)dP = \int g(r)dr\,.$$

We get the right answer with this method but please don't tell the mathematicians what we're doing. They hate us for this! The separation of the OV equation with our density profile is

$$\frac{dP}{(1 + P/\rho_0 c^2)\left(1 + 3P/\rho_0^2 c^2\right)} = -\frac{4\pi G\rho_0^2}{3}\frac{rdr}{(1 - 8\pi G\rho_0 r^2/3c^2)}\,.$$

The integration must be done from the outside in:

$$\int_{P(r)}^{0} \frac{dP'}{(1 + P'/\rho_0 c^2)(1 + 3P'/\rho_0 c^2)} = -\frac{4\pi G\rho_0^2}{3}\int_r^R \frac{r'dr'}{(1 - 8\pi G\rho_0 r'^2/3c^2)}\,.$$

A realistic treatment would look at the temperature to get the density and construct a stellar atmosphere but we do not worry about this.

$$\Rightarrow \qquad -\frac{1}{2}\rho_0 c^2 \left[\ln\left(\frac{1+P'/\rho_0 c^2}{1+3P'/\rho_0 c^2}\right)\right]_{P(r)}^{0} = \frac{1}{4}\rho_0 c^2 \left[\ln\left(1-\frac{8\pi G\rho_0}{3c^2}r'^2\right)\right]_r^R$$

$$\Rightarrow \qquad \frac{1+\frac{P(r)}{\rho_0 c^2}}{1+\frac{3P(r)}{\rho_0 c^2}} = \left(\frac{1-\frac{2GM}{c^2 R}}{1-\frac{2GM}{c^2 R}\frac{r^2}{R^2}}\right)^{1/2}$$

If we solve for $P(r)$ and rewrite $R_\mathrm{S} = 2GM/c^2$ then we get an equation for the pressure:

$$\boxed{P(r) = \rho_0 c^2 \left[\frac{\sqrt{1-R_\mathrm{S} r^2/R^3} - \sqrt{1-R_\mathrm{S}/R}}{3\sqrt{1-R_\mathrm{S}/R} - \sqrt{1-R_\mathrm{S} r^2/R^3}}\right] .} \tag{4}$$

Therefore, the central pressure is

$$\boxed{P_\mathrm{C} = P(r)|_{r=0} = \rho_0 c^2 \left(\frac{1-\sqrt{1-R_\mathrm{S}/R}}{3\sqrt{1-R_\mathrm{S}/R}-1}\right) .} \tag{5}$$

Recall the central pressure for the Newtonian case is

$$P_\mathrm{C} = \frac{2\pi}{3} G\rho_0^2 R^2 .$$

If we Taylor expand around small R_S/R we get the Newtonian case!
Notice: $P_\mathrm{C} \to \infty$ as $R \to \frac{9}{8}R_\mathrm{S}$.
Q: What does this mean? **A:** For every NS $R_\mathrm{NS} > \frac{9}{8}R_\mathrm{S}$!
Otherwise the pressure would be infinite. The argument is completely self-consistent and would work for softer (smoother) density profiles even if the exact calculation is different. However, this demands we are in HSE, so the breaking point is that we cannot maintain HSE if $R < \frac{9}{8}R_\mathrm{S}$.

Therefore we can derive an upper limit for the mass of NSs by equating two equations:

$$\text{(i)} \;\; R = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} M^{1/3} \qquad \text{and} \qquad \text{(ii)} \;\; \frac{9}{8}R_\mathrm{S} = \frac{9}{4}\frac{G}{c^2}M .$$

We now solve for the maximum mass of a Neutron Star:

$$\frac{9}{4}\frac{G}{c^2}M_\mathrm{max} = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} M_\mathrm{max}^{1/3}$$

and call it the "Oppenheimer–Volkoff Limit"

$$\boxed{M_\mathrm{OV} \equiv M_\mathrm{max} = \frac{8}{27}\left(\frac{c^2}{G}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} \approx 5\ M_\odot .} \tag{6}$$

Here we have assumed a high nuclear density of $\rho_0 \sim 5\times 10^{14}$ g/cm^3. Compare this to the Chandrasekhar limit for WDs of $M_\mathrm{Ch} \approx 1.4\ M_\odot$.
Note: Quantum Mechanics does not enter here! Also, we do not know the exact equation of state and for that reason we take M_OV as a rough upper limit.