AST 353 Astrophysics — Problem Set 2

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I. MASS AND DENSITY IN WHITE DWARFS

Assume a WD is supported by NR degenerate electrons. The mass-radius relation is

$$R \approx \frac{R_\odot}{100}\left(\frac{M}{M_\odot}\right)^{-1/3}. \tag{1}$$

(a) Central Pressure

To find the central pressure P_c we plug Eq. 1 into the formula derived in class:

$$\begin{aligned} P_c &\approx \frac{GM^2}{R^4} \qquad (2)\\ &= GM_\odot^2\left(\frac{M}{M_\odot}\right)^2\left[\left(\frac{100}{R_\odot}\right)^4\left(\frac{M}{M_\odot}\right)^{4/3}\right] \\ &= \frac{10^8 GM_\odot^2}{R_\odot^4}\left(\frac{M}{M_\odot}\right)^{10/3} \\ \Rightarrow \quad P_c &\approx K_1\left(\frac{M}{M_\odot}\right)^{x_1}. \qquad (3) \end{aligned}$$

If we use $G = 6.67\times10^{-8}$ cm^3/g/s^2, $M_\odot = 2\times10^{33}$ g, and $R_\odot = 7\times10^{10}$ cm then

$$K_1 = \frac{10^8 GM_\odot^2}{R_\odot^4} = \frac{10^8(6.67\times10^{-8}\ \text{cm}^3/\text{g}/\text{s}^2)(2\times10^{33}\ \text{g})^2}{(7\times10^{10}\ \text{cm})^4} = \boxed{1.1\times10^{24}\ \text{dyne}/\text{cm}^2}$$

$$\text{and} \quad \boxed{x_1 = 10/3\,.}$$

(b) Central density

WDs are supported by electron degeneracy pressure

$$\begin{aligned} P_c &\approx K_{\rm NR} n_e^{5/3} = K_{\rm NR}\left(\frac{\rho_c}{2m_H}\right)^{5/3} \\ \text{where} \quad K_{\rm NR} &= \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} = 2.3\times10^{-27}\ \text{cm}^4\ \text{g}/\text{s}^2\,. \qquad (4) \end{aligned}$$

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We have assumed two nucleons per electron. Also the particle rest mass should is for electrons because they provide the degeneracy pressure (i.e. $m_0 = m_e$). To find a mass-density relation we equate Eqs. 2 and 4:

$$\begin{aligned}
\rho_c &\approx 2m_H \left(\frac{P_c}{K_{\mathrm{NR}}}\right)^{3/5} \\
&= 2(1.67 \times 10^{-24}\ \mathrm{g}) \left[\frac{1.1 \times 10^{24}\ \mathrm{g/cm/s^2}}{2.3 \times 10^{-27}\ \mathrm{cm^4\ g/s^2}} \left(\frac{M}{M_\odot}\right)^{10/3}\right]^{3/5} \\
&= 8.5 \times 10^{6}\ \mathrm{g/cm^3} \left(\frac{M}{M_\odot}\right)^{2} \\
&= K_2 \left(\frac{M}{M_\odot}\right)^{x_2} . \quad \Rightarrow \quad \boxed{K_2 = 8.5 \times 10^{6}\ \mathrm{g/cm^3}} \quad \text{and} \quad \boxed{x_2 = 2\,.}
\end{aligned} \tag{5}$$

(c) Critical mass

If relativistic effects become important for densities $\rho_{\mathrm{crit}} \approx 10^5\ \mathrm{g/cm^3}$ then we can solve for the critical density from Eq. 5

$$M_{\mathrm{crit}} \approx \sqrt{\frac{\rho_{\mathrm{crit}}}{K_2}} M_\odot = \sqrt{\frac{10^5\ \mathrm{g/cm^3}}{8.5 \times 10^6\ \mathrm{g/cm^3}}} M_\odot = \boxed{0.11\ M_\odot\,.} \tag{6}$$

II. PLANCK LENGTH

Define the *Planck length* (l_{Pl}) as the Compton wavelength ($\lambda_{\mathrm{C}} = h/m_0 c$) corresponding to the *Planck mass* ($m_{\mathrm{Pl}} = \sqrt{hc/G}$) to find an algebraic and numeric length:

$$\begin{aligned}
l_{\mathrm{Pl}} &= \lambda_{\mathrm{C,Pl}} \\
&= \frac{h}{m_{\mathrm{Pl}} c} \\
&= \frac{h}{c}\sqrt{\frac{G}{hc}} \\
&= \sqrt{\frac{hG}{c^3}} \\
&= 4 \times 10^{-33}\ \mathrm{cm}\,.
\end{aligned} \tag{7}$$

This is the length scale where space itself is "quantized." Because of the uncertainty principle with position and momentum space becomes "foamy" and it is impossible to measure distances smaller than l_{Pl}. In the context of our definition of the Planck mass, the Planck length corresponds to the Schwarzschild radius R_{S} of a black hole with mass m_{Pl}:

$$R_{\mathrm{S,Pl}} \sim \frac{m_{\mathrm{Pl}} G}{c^2} = \sqrt{\frac{hc}{G}}\frac{G}{c^2} = \sqrt{\frac{hG}{c^3}} = l_{\mathrm{Pl}}\,.$$

Therefore, classical general relativity cannot properly describe gravity on scales smaller than l_{Pl}.