# AST 353 Astrophysics — Problem Set 1

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## I. SIMPLE STELLAR MODEL

Assume a star has a radius of $R = 3R_\odot$ and a quadratic density profile:

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right] = \rho_c \left(1 - x^2\right) .$$

Here, $\rho_c = 20$ g/cm$^3$ is the central density. We have defined a new variable $x = r/R$ to make future integrals easier, so the differential is $dr = Rdx$.

### (a) Mass

To find the total mass $M$ of the star we first find the mass coordinate $m(r)$, which is the mass interior to a shell of radius $r$:

$$\begin{aligned} m(r) &\equiv 4\pi \int_0^r r'^2 \rho(r') dr' \\ &= 4\pi \int_0^x (Rx')^2 \rho(x') (Rdx') \\ &= 4\pi \rho_c R^3 \int_0^x \left(x'^2 - x'^4\right) dx' \\ m(x) &= 4\pi \rho_c R^3 \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right) . \end{aligned}$$

The total mass $M$ is the mass inside a radius $R$, or where $x = 1$:

$$M = m(R) = 4\pi \rho_c R^3 \left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{8\pi}{15}\rho_c R^3 .}$$

We can find the solution in terms of solar masses by using the following information: $R = 3R_\odot$, $R_\odot = 7 \times 10^{10}$ cm, $\rho_c = 20$ g/cm$^3$, and $M_\odot = 2 \times 10^{33}$ g. Together this gives:

$$M = \frac{8\pi}{15}\left(20 \text{ g/cm}^3\right)\left(3 \cdot 7 \times 10^{10} \text{ cm}\right)^3 = 3.1 \times 10^{35} \text{ g} = \boxed{155\ M_\odot .}$$

### (b) Average density and free-fall time

Recall how averages are found:

$$\langle \rho \rangle = \frac{\int \rho\, dV}{\int dV} = \frac{1}{V}\int_0^R 4\pi r'^2 \rho(r') dr' = \frac{M}{V} = \frac{\frac{8\pi}{15}\rho_c R^3}{\frac{4\pi}{3}R^3} = \frac{2}{5}\rho_c = \boxed{8 \text{ g/cm}^3 .}$$

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The free-fall time $\tau_{\rm ff}$ is then (using $G = 6.67 \times 10^{-8}\ {\rm cm^3/s^2/g}$)

$$\tau_{\rm ff} \approx \frac{1}{\sqrt{G\langle\rho\rangle}} = \frac{1}{\sqrt{(6.67 \times 10^{-8}\ {\rm cm^3/s^2/g}) \cdot (8\ {\rm g/cm^3})}} = 1369\ {\rm s} = 22.8\ {\rm min} \approx \boxed{\frac{1}{2}\ {\rm hour}\,.}$$

**(c) Pressure**

For this we need the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho g = -\frac{G\rho(r)m(r)}{r^2}\,.$$

Both $\rho(r)$ and $m(r)$ depend on the radius!!! Finally we can think of this as a simple ODE with our boundary condition $P(R) = 0$ and the pressure is found by integration:

$$P(r) = P(r) - P(R) = P(r)\big|_R^r = \int_R^r \frac{dP(r')}{dr'}dr' = \int_r^R \frac{G\rho(r')m(r')}{r'^2}dr'\,.$$

In the final equality I switched the limits of integration because of the minus sign in the HSE equation. The integration is easier in terms of $x = r/R$:

$$\begin{aligned}
P(x) &= \int_x^1 \frac{G}{R^2x'^2}\rho_c\left(1 - x'^2\right)\left[4\pi\rho_c R^3\left(\frac{1}{3}x'^3 - \frac{1}{5}x'^5\right)\right](R dx') \\
&= \frac{4\pi G\rho_c^2R^2}{15}\int_x^1 \left(1 - x'^2\right)\left(5x' - 3x'^3\right)dx' \\
&= \frac{4\pi G\rho_c^2R^2}{15}\int_x^1 \left(5x' - 8x'^3 + 3x'^5\right)dx' \\
P(x) &= \boxed{\frac{4\pi G\rho_c^2R^2}{15}\left(1 - \frac{5}{2}x^2 + 2x^4 - \frac{1}{2}x^6\right)\,.}
\end{aligned}$$

**(d) Central pressure**

The central pressure is simply the pressure at $r = x = 0$

$$P_c = \frac{4\pi G\rho_c^2R^2}{15} = \frac{4\pi}{15}(6.67\times10^{-8}\ {\rm cm^3/s^2/g})\left(20\ {\rm g/cm^3}\right)^2\left(3\cdot 7 \times 10^{10}\ {\rm cm}\right)^2 = \boxed{9.86 \times 10^{17}\ {\rm dyne/cm^2}\,.}$$

We may always check our answer by the OoMA estimate given in class

$$P_c \approx \frac{GM^2}{R^4} = (6.67 \times 10^{-8}\ {\rm cm^3/s^2/g})\,(3.1 \times 10^{35}\ {\rm g})^2\left(3\cdot 7 \times 10^{10}\ {\rm cm}\right)^{-4} = \boxed{3.3 \times 10^{18}\ {\rm dyne/cm^2}\,.}$$

These two methods are the same order of magnitude ($\sim 10^{18}\ {\rm dyne/cm^2}$) so we know we have the right answer!

**(e) Total gravitational potential energy**

The gravitational potential energy is given by

$$E_{\rm pot} = -\int_0^M \frac{Gm(r)}{r}dm\,.$$

However, the mass $m(x)$ in terms of $M$ is

$$m(x) = Mx^3\left(\frac{5}{2} - \frac{3}{2}x^2\right)$$

and the mass differential $dm$ is given in terms of $M$ is

$$dm = 4\pi r^2\rho(r)dr = 4\pi R^3 x^2 \rho(x)dx = \frac{15}{2}Mx^2\left(x^2-1\right)dx\,,$$

so the total gravitational potential energy is given by

$$\begin{aligned}
E_{\text{pot}} &= -\int_0^M \frac{Gm(r)}{r}dm \\
&= -\int_0^1 \frac{G}{Rx}\left[Mx^3\left(\frac{5}{2} - \frac{3}{2}x^2\right)\right]\left[\frac{15}{2}Mx^2\left(x^2-1\right)dx\right] \\
&= -\frac{GM^2}{R}\int_0^1\left(\frac{75}{4}x'^4 - 30x'^6 + \frac{45}{4}x'^8\right)dx \\
&= \boxed{-\frac{5}{7}\frac{GM^2}{R}\,.}
\end{aligned}$$

These quantities can be evaluated to give the energy in ergs

$$|E_{\text{pot}}| = \frac{5}{7}\frac{GM^2}{R} = \frac{5}{7}\frac{(6.67\times10^{-8}\ \text{cm}^3/\text{s}^2/\text{g})\,(3.1\times10^{35}\ \text{g})^2}{(3\cdot 7\times10^{10}\ \text{cm})} = \boxed{2\times10^{52}\ \text{erg}\,.}$$

## II. PARTICLE KINETIC ENERGIES

From Special Relativity we know a particle's total energy $\epsilon$ is related to its momentum $p$ and rest mass $m_0$ by

$$\epsilon^2 = p^2c^2 + m_0^2c^4\,.$$

If we define the kinetic energy as

$$\epsilon_{\text{kin}} = \epsilon - m_0c^2$$

then using a Taylor expansion ($\sqrt{1+x} \approx 1 + x/2$ if $x$ is small) and using the defining property for NR particles that $pc \ll m_0c^2$ then the kinetic energy reduces to the NR version:

$$\begin{aligned}
\epsilon_{\text{kin}} &= \epsilon - m_0c^2 \\
&= \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 \\
&= m_0c^2\sqrt{1 + \left(\frac{pc}{m_0c^2}\right)^2} - m_0c^2 \\
&\approx m_0c^2\left[1 + \frac{1}{2}\left(\frac{pc}{m_0c^2}\right)^2\right] - m_0c^2 \\
&= \frac{m_0c^2}{2}\left(\frac{p^2c^2}{m_0^2c^4}\right) \\
&= \frac{p^2}{2m_0} \quad \left(\text{or } \frac{1}{2}m_0v^2\right)\,.
\end{aligned}$$