AST 353 Astrophysics — Exam 1 Solutions

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I. VIRIAL EQUILIBRIUM (4 POINTS)

Consider a gas cloud comprised of hydrogen. The mass $M = 10^6\ M_\odot$, radius $R = 50$ pc, and temperature $T = 10$ K. The energy from the gas comes from assuming an ideal gas law:

$$E_{\rm kin} \approx N k_{\rm B} T \approx \frac{M}{m_{\rm H}} k_{\rm B} T \approx \frac{10^6 \times 2 \times 10^{33}\ \text{g}}{1.67 \times 10^{-24}\ \text{g}} (1.38 \times 10^{-16}\ \text{erg K}^{-1})(10\ \text{K}) \approx 10^{48}\ \text{erg}\,.$$

We compare this to the energy required to keep it in virial equilibrium:

$$|E_{\rm pot}| \approx \frac{GM^2}{R} \approx \frac{(6.67 \times 10^{-8}\ \text{erg cm/g}^2)(10^6 \times 2 \times 10^{33}\ \text{g})^2}{50(3.09 \times 10^{18}\ \text{cm})} \approx 10^{51}\ \text{erg}\,.$$

The cloud is $\boxed{\text{NOT}}$ in virial equilibrium because the gas has 1000 times the required energy.

$$\boxed{P_{\rm kin} \ll P_{\rm pot} \quad \Rightarrow \quad \text{The radius will decrease and start to collapse the cloud.}}$$

II. SIMPLE STELLAR MODEL (8 POINTS)

(a) What is the mass M of the star (in units of $M_\odot$)?

A star has a radius of $R = 2R_\odot$ and a uniform (constant) density profile: $\rho(r) = \rho_0 = 50\ \text{g/cm}^3$. The mass coordinate $m(r)$ gives the mass contained in a radius r (recall that $dm = 4\pi r^2 \rho dr$)

$$m(r) = 4\pi \int_0^r r'^2 \rho(r') dr' = 4\pi\rho_0 \int_0^r r'^2 dr = \frac{4}{3}\pi r^3 \rho_0\,.$$

Therefore, the total mass M is

$$M = m(2R_\odot) = \frac{4}{3}\pi \left(2 \times 6.96 \times 10^{10}\ \text{cm}\right)^3 \left(50\ \text{g/cm}^3\right) = 5.65 \times 10^{35}\ \text{g} = \boxed{282.5\ M_\odot\,.}$$

(b) What is the free-fall time $\tau_{\rm ff}$ for this star, in suitable units (s, h, years...)?

The free-fall time $\tau_{\rm ff}$ is the time it takes a star to collapse if its central pressure engine was suddenly turned off. We use the estimate

$$\tau_{\rm ff} \approx 1/\sqrt{G\rho_0} \approx \frac{1}{\sqrt{(6.67 \times 10^{-8}\ \text{cm}^3/\text{g/s}^2)(50\ \text{g/cm}^3)}} \approx 548\ \text{s} \approx \boxed{9\ \text{min}\,.}$$

(c) Find the pressure $P(r)$

To do this we solve the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho \frac{Gm(r)}{r^2} = -\frac{4}{3}\pi G \rho_0^2 r\,,$$

assuming the *zero boundary condition* so $P(R) = 0$. We can directly integrate the previous equation:

$$P(r) = \int_R^r \frac{dP}{dr'} dr' = \frac{4}{3}\pi G \rho_0^2 \int_r^R r' dr' = \left[\frac{2}{3}\pi G \rho_0^2 r'^2\right]_r^R = \frac{2}{3}\pi G \rho_0^2 R^2 \left[1 - \left(\frac{r}{R}\right)^2\right] .$$

If $x = r/R$ is a dimensionless radius then we can put the pressure in the form:

$$P(r) = K\ [1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots] , \text{ where } \boxed{K = \frac{2}{3}\pi G \rho_0^2 R^2,\ a_2 = -1,\ \text{and all other } a_i = 0 .}$$

(d) Find the central pressure P_c

We can use the result of (c) to evaluate the central pressure P_c:

$$\begin{aligned} P_c = P(r = 0) = K &= \frac{2}{3}\pi G \rho_0^2 R^2 \\ &= \frac{2}{3}\pi \left(6.67 \times 10^{-8}\ \text{dyn cm}^2/\text{g}^2\right) \left(50\ \text{g/cm}^3\right)^2 \left(2 \times 6.96 \times 10^{10}\ \text{cm}\right)^2 \\ &= \boxed{6.77 \times 10^{18}\ \text{dyn/cm}^2 .} \end{aligned}$$

We confirm that this value is close to the estimate

$$P_{c,\text{est}} \approx \frac{GM^2}{R^4} = \frac{\left(6.67 \times 10^{-8}\ \text{dyn cm}^2/\text{g}^2\right) \left(5.65 \times 10^{35}\ \text{g}\right)^2}{\left(2 \times 6.96 \times 10^{10}\ \text{cm}\right)^4} = \boxed{5.67 \times 10^{19}\ \text{dyn/cm}^2 .}$$

We usually expect P_c to be smaller than $P_{c,\text{est}}$ by a factor of order unity. This is small but ok.

III. PLANCK TIME (2 POINTS)

There are more sophisticated ways of doing this but one way is to define the *Planck time* (t_{Pl}) in terms of the *Planck length* ($l_{\text{Pl}} = \sqrt{hG/c^3}$):

$$t_{\text{Pl}} = \frac{l_{\text{Pl}}}{c} = \boxed{\sqrt{\frac{hG}{c^5}}} = \sqrt{\frac{(6.626 \times 10^{-27}\ \text{cm g/s})(6.67 \times 10^{-8}\ \text{cm}^3/\text{g/s}^2)}{(3 \times 10^{10}\ \text{cm/s})^5}} \approx \boxed{10^{-43}\ \text{s} .}$$

This is the scale where time is "quantized." Because of the uncertainty principle with time and energy 'spacetime' becomes "foamy" and we cannot measure times smaller than t_{Pl}. In the context of our definition of the Planck mass, the Planck time corresponds to the crossing time of the smallest classical black hole (i.e. with mass m_{Pl}).

$$t_{\text{BH}} = \frac{R_{\text{S,Pl}}}{c} \sim \frac{m_{\text{Pl}} G}{c^3} = \sqrt{\frac{hc}{G}} \frac{G}{c^3} = \sqrt{\frac{hG}{c^5}} = t_{\text{Pl}} .$$

As long as we maintain *causality* classical GR cannot apply to time scales smaller than t_{Pl}. Therefore the *Planck time* is also the shortest possible free-fall time (i.e. $\tau_{\text{ff,BH}} \sim \sqrt{l_{\text{Pl}}^3 / G m_{\text{Pl}}} = t_{\text{Pl}}$). Cosmologically, we need quantum gravity at times $t < t_{\text{Pl}}$ after the Big Bang.

IV. CHANDRASEKHAR MASS (1 POINT)

The *Chandrasekhar mass* is the largest stable mass of a star supported by electron degeneracy pressure. This is important in astrophysics as it determines the end state of many stars. Stars less massive than $1.4\ M_\odot$ remain stable and will cool over time. However, for stars more massive than this gravitational collapse can lead to supernova explosions, neutron stars, or even black holes.