Black Holes

Early speculations: "Dark Star"

In late 18th century (Enlightenment)
- John Michell (1783)
- Pierre Simon Laplace (Le Système du Monde, 1786)

→ 2 ingredients

(1) Newtonian gravity

(2) Corpuscular theory of light (also Newtonian)

\[ E_{\text{tot}} = -\frac{GMm}{R} + \frac{1}{2} mv^2 \]

For no escape, need \( E_{\text{tot}} \leq 0 \)

\[ v > v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \]

For particle to escape

For light, \( v_{\text{esc}} = c \), a critical radius:

\[ R = \frac{2GM}{c^2} \]
Later called the Schwarzschild radius.

- Crucial difference between dark stars and modern black holes.

→ For the dark star, close-in observers would still be able to see the object.

→ For modern black holes, photons cannot travel beyond \( r > R_s \).

Schwarzschild Geometry

- Use GR to consider geometry (or gravitational field) outside of a spherically symmetric object.

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

- Choose, for convenience, spherical coordinates \((r, \theta, \phi)\), reflecting the spherical symmetry.
Space-time must have general form:

\[ ds^2 = -A(r) c^2 dt^2 + B(r) dr^2 + r^2 (\sin^2 \theta d\theta^2 + d\theta^2) \]

Find solution to Einstein field equation outside of the spherical mass distribution ("vacuum solution")

\[ G_{uv} = \frac{8\pi G}{c^4} T_{uv} \]

\[ G_{uv} \left( \frac{\partial^2 \delta_{uv}}{\partial x^a \partial x^b} + \frac{\partial^2 \delta_{uv}}{\partial x^b \partial x^a} - g_{uv} \frac{\partial^2 \delta_{ab}}{\partial x^a \partial x^b} \right) = 0 \]

Find solution

\[ A(r) = g_{oo} = 1 - \frac{2GM}{c^2 r} \]

\[ B(r) = g_{11} = \frac{1}{A(r)} \]
\[ ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 \left(\sin^2 \theta d\theta^2 + d\phi^2\right) \]

"Schwarzschild metric" (K. Schwarzschild, 1916)

- Interpretation of radial coordinate:
  - *NOT* the physical distance from the center → space-time is curved
  - "Circumferential radius"

- Consider the physical (proper) area on a surface with \( r = \text{const.} \)
  \[ A = r^2 \int_0^{2\pi} \sin \theta d\theta d\phi = 4\pi r^2 \]
  \[ C = 2\pi r \]

Flow of time in Schwarzschild geometry

Consider 2 stationary observers \( dr = d\theta = d\phi = 0 \)

(A) at \( r \to \infty \)

(B) at \( r \)
- Figure out proper time that each observer would measure, \( \tau \)

\[ -c^2 d\tau^2 = ds^2 \]

(A) \( r \to \infty \Rightarrow ds^2 = -c^2 dt^2 = -c^2 d\tau^2 \)

Thus, \( d\tau = dt \)

\( \text{coordinate time} \)

(B) \( ds^2 = -(1 - \frac{2GM}{c^2r})c^2 dt^2 = -c^2 d\tau^2 \)

\( \Rightarrow d\tau = \left(1 - \frac{2GM}{c^2r}\right)^{\frac{1}{2}} dt \)

\[ d\tau = \left(1 - \frac{2GM}{c^2r}\right)^{\frac{1}{2}} d\tau \]

- Time slows down deep inside a gravitational potential well ("gravitational time dilation")

- Effect on light

(B) emits light at frequency \( \frac{\nu_m}{\tau} \)
(A) measure different frequency, $\nu = \frac{1}{\Delta \omega}$

$$\Rightarrow \nu = \nu_m \left(1 - \frac{2GM}{c^2 r}\right)^{\frac{1}{2}}$$

("gravitational redshift")

- Photons emitted at $r=R_c$ are redshifted to $\nu \rightarrow 0$ as $r \rightarrow \infty$. No photons make it to $r \rightarrow \infty$!