White Dwarfs (cont)

Rephrase the Chandrasekhar mass limit for a white dwarf in terms of the 'Plancher mass,'

\[ m_{\text{pl}} = \left( \frac{hc}{g} \right)^{\frac{1}{2}} = 5 \times 10^{-5} \text{ g} \]

\[ = 10^{19} \frac{\text{GeV}}{c^2} \]

meaning: mass of the smallest black hole that can be described by GR

Compton wavelength, a fundamental distance, \( \lambda_c \), from a mass at rest

\[ E = m_c c^2 = \frac{hc}{\lambda_c} \]

For a black hole to be well-defined, need

\[ GR \rightarrow R_s \geq \lambda_c \]

\[ \Rightarrow \frac{G M_{\text{BH}}}{c^2} \geq \frac{h}{m_c c^2} \]

\[ \Rightarrow M_{\text{BH}} \geq \left( \frac{h c}{G} \right)^{\frac{1}{2}} \]
At the Planck scale, we need quantum gravity to describe nature \((G_R + QM)\)

\[
\Rightarrow M_{Pl} = \frac{1}{4} \left( \frac{3}{8} \right)^{3/2} \frac{m_p^3}{m^2} \\
\Rightarrow M_{Pl} = \frac{m_p^3}{m^2} \\
\]

Now, rephrase \(M_{Pl}\) in terms of the strength of gravity:

Recall how to measure strength of an E-M interaction:

\[
U_{em} = \frac{1}{4\pi \epsilon_0} \frac{e^2}{r} \\
\]

Evaluate \(U_{em}\) at \(r = \lambda_c\). This defines the "fine structure constant":

\[
\lambda = \left( \frac{U_{em}(r = \lambda_c)}{M_H c^2} \right) = \frac{1}{4\pi \epsilon_0} \frac{e^2}{\lambda_c c^2} = \frac{1}{\frac{4\pi \epsilon_0}{\lambda_c c^2} \frac{e^2}{M_H c^2}} = \frac{e^2}{4\pi \epsilon_0 \lambda_c} = \frac{1}{137}
\]
By analogy, define the "gravitational time structure constant":

$$\xi_G = \left| \frac{U_g (r = \lambda c)}{m_H c^2} \right| = \frac{G m_H^2}{\lambda c} = 10^{-38},$$

much smaller than \( \xi \), i.e. gravity is very weak compared to electromagnetism.

Now, \( M_{ch} = \left( \frac{hc}{G} \right)^{3/2} \frac{1}{m_H} = \left( \frac{hc}{GM_H} \right)^{3/2} m_H \)

$$= \xi_G^{3/2} m_H$$

$$= 10^{57} m_H \approx 1 M_\odot$$

Since gravity is so weak we need \(-10^{57}\) nucleons to overpower the e.m. interaction.

\( \Rightarrow M_{ch} \approx 1 M_\odot \), the typical mass of any star in our universe.

**Cooling of White Dwarfs**

\[ L = \frac{\text{energy}}{\text{time}} \]

\[ R = \text{radius} \]

\[ T = \text{surface temperature} \]
\[ L = 4\pi R^2 \sigma_{SB} T_{eff}^4 \]  \hspace{1cm} \text{(Stefan-Boltzmann Law)}

\[ \sigma_{SB} = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \]

- Mass-radius relation for a NR WD:

\[ R = 0.01 \text{ R}_0 \left( \frac{M}{M_0} \right)^{-\frac{1}{3}} \]

From observations, we know that \( T_{eff} = 20,000 \text{ K} \)

\[ \Rightarrow L_{WD} = 10^{-2} L_0 \left( \frac{M}{M_0} \right)^{-\frac{1}{3}} \left( \frac{T_{eff}}{20,000} \right)^4 \]

Consider cooling tracks of WDs in the Hertzsprung-Russel diagram:

- Q: How long does it take for a WD to lose all of its thermal energy, \( E_{th} \)?

A: Define the 'cooling time':

\[ \tau = \frac{E_{th}}{L_{WD}} \]

\[ L_{WD} = 10^{-2} L_0 \]

\[ E_{th} = 4 \times 10^{26} \text{ W} \]
\[ E_{\text{thrm}} = \frac{3}{2} k_B T \frac{M}{12 \, \text{m}_H} \]

For WD: \[ M = 1 \, M_\odot \]
\[ T = 10^8 \, \text{K} \] (\( 3\text{He}^4 \rightarrow \text{C}^{12} \))

\[ \Rightarrow E_{\text{th}} = 10^{41} \, \text{J} \]

\[ \Rightarrow T_{\text{cool}} = 10^9 \, \text{yr} \] - comparable to the age of the universe/galaxy

\[ t_H = 10^{10} \, \text{yr} \]