**Four Dimensions**

- Space (Cartesian) coordinate: \((x,y,z)\)
  - Distance\(^2 = (x \text{ interval})^2 + (y \text{ interval})^2 + (z \text{ interval})^2\)
- Fourth dimension is **time**
  - More general coordinate = **spacetime**: \((x,y,z,t)\)
  - \((\text{Spacetime distance})^2\)
    \[= c^2(t \text{ interval})^2 - [(x \text{ interval})^2 + (y \text{ interval})^2 + (z \text{ interval})^2]\]
- **Spacetime Diagram**
  - “Events” connected by “Worldlines”

**Spacetime Distances**

\[(\text{Spacetime distance})^2 = c^2(t \text{ interval})^2 - (x \text{ interval})^2 - (y \text{ interval})^2 - (z \text{ interval})^2\]

**Velocity and Light cone**

- The steeper the slope is, the smaller the velocity is.
- The lines with 45 degrees tilt represent the “light cone”
- Since nothing can travel faster than light…
  - O and A can communicate, but O and B cannot communicate
  - This diagram shows a “causal structure”

**Acceleration and Deceleration**

- OA: Decelerated
- OB: Constant velocity
- OC: Accelerated
Special Relativity (1905)

- Two Invariants
  - Speed of light, \( c \)
  - Spacetime distance \( (ds^2 = c^2 dt^2 - dx^2) \)
- Unification of space and time
  - No absolute space or time exists: Relativity
- Special relativity does not include gravity

Relativity of Space and Time

- A’s space coordinate, \( x \), does not coincide with B’s, \( x' \). Rather, \( x \) is a combination of \( x' \) and \( ct' \).
- The same is true for time coordinate.
  - This means that simultaneous events in A’s coordinate would not appear simultaneous in B’s coordinate.
- But, spacetime distance remains unchanged.

Time Dilation and Length Contraction

- When A sees B moving, B’s time interval appears to be longer (clock ticks more slowly; time dilation) and B’s length appears to be shorter (length contraction). And vice versa.

Intuitive way to understand it

- From your point of view, the ball appears to move faster; however, light cannot travel faster!
- Therefore, it must take light more time to come back down to the laser – Time Dilation
Relativistic Gamma Factor

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

- \( \gamma \) is always greater than 1.
- As \( v \) approaches \( c \), \( \gamma \) becomes large.
- When \( v = c \), \( \gamma \) is infinite.

- B’s unit time in A’s frame equals A’s unit time in A’s frame multiplied by \( \gamma \). (Hence time dilation)
  - Be careful! The time actually elapsed in B’s frame gets shorter because the unit time gets longer.
- B’s unit length in A’s frame equals B’s unit length in B’s frame divided by \( \gamma \). (Hence length contraction)

Mass Increase

- A pushes B (whose mass at rest is \( m \)) by applying a force \( F \).
  - Acceleration is given by \( a = F/m \).
  - Velocity acquired would be \( v = a \ dt = F \ dt / m \).
- When B is moving, the clock ticks more slowly
  - B feels the force for a shorter time
    - \( v' = a \ dt' = F \ dt' / m = F \ dt / (m \gamma) \)
- Thus, the mass of B appears to be bigger by \( \gamma \).
- Nothing can be accelerated to the speed of light, because the mass becomes infinite.

Twin Paradox

- There are twins, A and B
- B moves relative to A
  - A’s point of view
    - B is moving at speed \( v \)
    - B’s clock ticks more slowly by \( \gamma \).
    - Therefore, B appears to be aging more slowly.
  - B’s point of view
    - A is moving at speed \( v \)
    - A’s clock ticks more slowly by \( \gamma \).
    - Therefore, A appears to be aging more slowly.
- So, which one is older, when they meet?
  - Twin Paradox

Case 1

- A and B are at rest at the same place until event 1.
- Then A and B go on a trip on opposite directions.
- A and B turn around and come back at events 2.
- A and B finally meet at event 3.
- In this case, A’s and B’s worldlines are symmetric.
- A and B have traveled the same spacetime distance.
  - Therefore, A and B have aged the same years.
Case 1 (a different point of view)

- C’s point of view
  - C is moving to the left with respect to the original frame
- A and B are moving to the right together until event 1.
- Then A is at rest but B speeds up.
- A turns around earlier than B.
  - Then B is at rest but A moves to the right faster than before
- A and B finally meet at event 3.
- In this case, A’s and B’s worldlines are still symmetric.
  - A and B have traveled the same spacetime distance; thus, A and B have aged the same years.

Case 2

- A remains at rest at all times.
- B leaves home at event 1, turns around at event 2, and finally meets A at event 3.
- In this case, A’s and B’s worldlines are not symmetric!
- What happens?
  - The answer is that A has aged more than B.
- Why?
  - B’s spacetime distance is shorter than A’s
  - Remember, \( ds^2 = c^2 dt^2 - dx^2 \)

So, what was it?

- Motion of A and B remains completely relative only when both are moving at constant velocity.
  - Motion has to be inertial for two frames to be completely equivalent.
- However, for two people to know their initial ages and then meet later again, the motion cannot stay inertial → two frames are no longer equivalent.