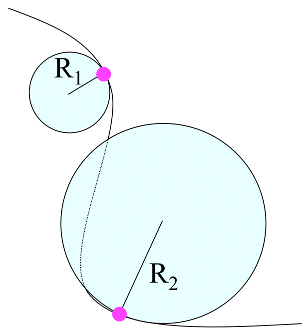
Curvature

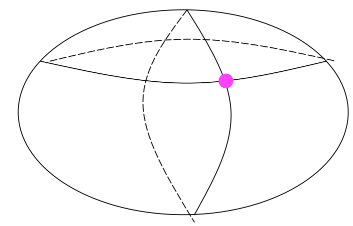
- How curved is it?
  - The radius of an *osculating circle* can be used to measure curvature of a line at a given point.
  - Curvature = 1/(curvature radius)
    - Curvature is in units of 1/length
  - The signs posted on the road saying “R=300ft” or “R=500ft”
    - Which one is more curved?
- A straight line (zero curvature) has R=infinity



Gaussian Curvature

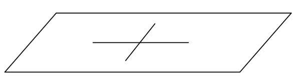
- Curvature of a surface
  - Draw **two** principal osculating circles at a given point on the surface
  - Obtain two principal curvature radii, R_1 and R_2
  - Gaussian curvature is given by $1/(R_1 R_2)$, **up to the overall sign.**
- K=Gaussian curvature
  - K is in units of 1/area

Johann Carl Friedrich Gauss (1777-1855)

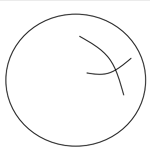


Flat, Spherical, Hyperbolic

- Homogeneous and Isotropic space can be either flat, spherical, or hyperbolic.
  - ***K* is the same everywhere**
- Flat
  - Zero K
  - K=0
- Spherical
  - Positive K
  - $K=1/R^2 > 0$
- Hyperbolic
  - Negative K
  - $K= -1/R^2 < 0$

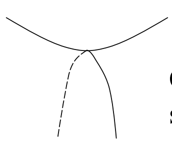


$R_1 = R_2$=Infinity



$R_1 = R_2$=R

Osculating circles on the same side



$R_1 = R_2$=R

Osculating circles on opposite sides

Measuring Curvature

- θ=Sum of the angles of a triangle minus π
  - **θ=*K* x (area of triangle)**
  - θ=0 (flat)
  - θ>0 (spherical)
  - θ<0 (hyperbolic)
- θ=Change in direction of an arrow through a closed circuit of the “vector transport”
  - **θ=*K* x (area enclosed by circuit)**
  - This is neat – try it yourself!

Distances in Curved Space

- In flat space (Euclidean space), a distance between two points on a surface is given by
  - **(Distance)^2 = (x-int.)^2 + (y-int.)^2**
  - $ds^2 = dx^2 + dy^2$
- In curved space (non-Euclidean), Euclidean formula no longer applies:
  - **(Distance)^2**
    = F(x-int.)^2 + 2G(x-int.)(y-int.) + H(y-int.)^2
  - $ds^2 = \mathrm{F}dx^2 + 2\mathrm{G}dxdy + \mathrm{H}dy^2$
  - F, G, H : *Metric Coefficients*
    - Metric coefficients generally depend on x and y

Gauss's Theorema Egregium

- Gaussian curvature is an "intrinsic" one.
  - We don't need to know anything about 3rd dimension to measure Gaussian curvature of 2-dimensional space
    - "Flatlanders" can measure curvature of their world (which is 2 dimensional) without knowing anything about the 3rd dimension.
  - Therefore, we can measure Gaussian curvature of 3-dimensional space without knowing anything about 4th dimension.
- ***Theorema Egregium***
  - How metric coefficients vary from point to point on a surface contains **all the information of the geometry** of the surface.

Riemannian Geometry

- Riemann has extended Gauss's work to four and higher dimensions.
- Metric coefficients have:
  - 3 components in 2d
  - 6 components in 3d
  - 10 components in 10d
  - etc…
- **Riemann curvature**: generalization of Gaussian curvature in higher dimensions.
- Einstein used Riemannian geometry to construct the general theory of relativity.

Georg Friedrich Bernhard Riemann (1826-1866)

Spacetime Metric of the Universe

$$ds^2 = c^2dt^2 - \left[\frac{dR^2}{1-KR^2} + R^2\left(d\alpha^2 + \sin^2\alpha d\theta^2\right)\right]$$

- Important question: **What is *K*** of the universe?
  - K determines curvature of **3-d space** in which we are living
- According to Gauss's theorema egregium, we can measure K, without knowing anything about the 4th dimension.
  - This implies that the shape of the observable universe can be determined!